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Edited by

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All communications regarding advertisements must be addressed to the Editor.

Remittances of membership fees and of subscriptions should be sent to the Treasurer, Prof. L. N. SUBRAMANIAM, Department of Mathematics, Christian College, Tambaram, Madras Presidency.

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TWELFTH CONFERENCE OF THE INDIAN MATHEMATICAL SOCIETY, HELD AT ALIGARH, ON THE 27th, 28th & 30th DECEMBER 1941.

Sitting :—

C. P. S. Menon (*Dehra Dun*), S. M. Shah (*Aligarh*), C. N. Srinivasiengar (*Bangalore*), A. N. Singh (*Lucknow*), F. W. Levi (*Calcutta*), Sir Ziauddin Ahmad (*Aligarh*)

R. Vaidyanathaswamy (*Madras*), A. B. A. Haleem (*Aligarh*), Khan Bahadur A. M. Kureishy (*Aligarh*), S. W. Shiveshwarkar (*Lucknow*)

A. Narasinga Rao (*Annamalainagar*), Ram Behari (*Delhi*), D. D. Kapadia (*Poona*), M. M. Sharif (*Aligarh*), S. M. Kerawala (*Aligarh*), Mohammad Ozair (*Aligarh*)

Standing 1st Row :—

Sahib Ram (*Lahore*), Hansraj Gupta (*Hoshiarpur*), S. R. Das (*Delhi*), B. R. Pasricha (*Lucknow*), Omar Ali Siddiqi (*Aligarh*), V. Rangachariar (*Patna*)

P. D. Shukla (*Lucknow*), R. S. Varma (*Lucknow*), R. D. Misra (*Lucknow*), B. Ramamurti (*Ajmer*), R. Krishnamurti (*Hyderabad Dn.*), Ekanath Banerji (*Cawnpore*)

S. C. Sircar (*Bombay*), D. N. Moghe (*Bombay*), G. Sarwar (*Aligarh*), M. Ziauddin (*Lahore*), D. J. Madan (*Karachi*), S. K. Abhyankar (*Gwalior*), Abdullah Butt (*Aligarh*)

Abdul Jabbar (*Calcutta*), M. Zakiuddin (*Aligarh*)

Standing 2nd Row :-

S. S. Pillai (*Travancore*), Ram Ballabh (*Lucknow*), Ahmad Abbas (*Hyderabad Dn.*), Prithvi Nath Sharma (*Delhi*), A. Vindhyachal Prasad (*Chapra*), S. Nageswaran (*Delhi*)

D. R. Kaprekar (*Devlali*), G. R. Seth (*Delhi*), F. C. Auluck (*Lucknow*), G. S. Diwan (*Bombay*), S. M. Shafi (*Aligarh*), Tahir Rizir (*Aligarh*), Zakiuddin, (*Aligarh*)

Hari Shanker (*Delhi*), D. D. Kosambi (*Poona*), K. R. Gunjikar (*Bombay*), Banwari Lal (*Meerut*), P. B. Bhattacharya (*Delhi*)

Standing 3rd Row :-

Nazir Ahmad (*Delhi*), Hashim (*Peon*), Abdul Hafeez (*Aligarh*), P. C. Mital (*Lucknow*), P. N. Saxena (*Lucknow*), Afzal Ahmad (*Hyderabad Dn.*), Misbahauddin Khan (*Aligarh*)

M. L. Widhani (*Aligarh*), Mohsin Siddiqi (*Aligarh*), Mohd. Shabbar (*Aligarh*), Mufizul Islam (*Aligarh*), K. Ramachandran (*Hyderabad Dn.*)

THE MATHEMATICS STUDENT

Volume X] MARCH 1942 [Number 1

THE TWELFTH CONFERENCE

OF THE

INDIAN MATHEMATICAL SOCIETY

The Indian Mathematical Society held its Twelfth Conference at Aligarh at the invitation of the Muslim University on the 27th, 28th and 30th of December 1941.

The Conference opened at 11 A.M. on Saturday the 27th December in the Strachey Hall. The delegates were welcomed by Prof. A. B. A. Haleem, Pro-Vice-Chancellor, and the Conference was inaugurated by Sir Ziauddin Ahmad, Vice-Chancellor of the Muslim University, Aligarh. Dr. Ram Behari (Delhi), the Secretary of the Society then read the Report of the Society's activities, and this was followed by the Presidential Address by Dr. R. Vaidyanathaswami (Madras) President of the Society.

Besides the reading of the papers, on the 28th and 30th December, there were three symposia—one on "Fourier integrals and transforms" with Dr. R. S. Varma (Lucknow) presiding, a second on "Group Theory" under the Chairmanship of Dr. F. W. Levi (Calcutta) and a third on "The Origin of the Solar System" initiated and presided over by Prof. A. C. Banerji (Allahabad).

There were 2 evening lectures, one by Dr. A. Narasinga Rao (Annamalainagar) on "Mathematics and Modern Warfare" and the other by Dr. B. Ramamurti (Ajmer) on "How to make the teaching of Mathematics interesting" Prof. D. D. Kosambi (Poona) addressed the Students of the Mathematics Association of the Muslim University on the evening of the 27th December.

As usual there was a Business Meeting of the Society and a Discussion on *the Teaching of Mathematics in Schools and Colleges*, initiated by Sir Ziauddin Ahmad.

While the serious side of the Conference thus received due attention, the social side was by no means neglected. The guests were comfortably lodged and fed more sumptuously than was good for their health. Specially worthy of mention were the splendid Tea on the 27th and Lunch on the 30th to which the guests were invited by Sir Ziauddin Ahmad and the rich Dinner on the 28th provided by Seth Phoolchandji of Aligarh City.

Special features which made the Aligarh conference noteworthy were the *Annual Session of the Benares Mathematical Society* held for the first time

in collaboration with the Indian Mathematical Conference, and the announcement of a prize to be awarded once in two years for the encouragement of Mathematical research which Prof. A. Narasinga Rao offered to institute at his own expense. The details of the prize are under consideration by the Committee of the Indian Mathematical Society.

Reception Committee

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Lt.-Col. Dr. Sir Ziauddin Ahmad, M.A., Ph.D., D.Sc., Kt., C.I.E., M.L.A. (Central), Vice-Chancellor, Muslim University, Aligarh.

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Welcome Address

BY

PROF. A. B. A. HALEEM, *Pro-Vice-Chancellor*

(27th December 1941)

Mr. President and Delegates to the Indian Mathematical Conference:

I deem it a very pleasant duty and a great privilege to welcome on behalf of the Aligarh Muslim University the members of the Indian Mathematical Conference. The invitation to the Conference was extended in fulfilment of the wish expressed by the late Sir Shah Mohammad Sulaiman, the then Vice-Chancellor of the University, and it is a matter of grief to us all that that distinguished scientist is no longer in our midst. The place vacated by the death of Sir

Shah at the early age of 54 was filled up by Sir Ziauddin Ahmad, who has won distinction and fame in various fields of activity, including the field of work to which the members of this Conference have dedicated themselves. It is but seldom in the life even of an educational institution like ours that such a gathering of the leading Mathematicians of India takes place, and we are grateful to the President and the Secretary of the Conference for having kindly accepted our invitation thus enabling us to establish a contact between the delegates on the one hand and the staff and students of this University on the other.

The University to which I am welcoming you today was founded with the object of combining the study of Western Arts and Sciences with Eastern learning. Though called into being for the purpose of meeting the requirements of one of the two major peoples of this country, it has always managed to attract students belonging to all creeds and faiths in large numbers. Drawing its alumni from all the provinces of India and even from foregin lands, it has aimed, not without a large measure of success, at developing a broad outlook on life and a spirit of large-hearted toleration. The study of the science of Mathematics has always been duly stressed at this University and our Mathematics Department has been staffed by a succession of able men. The first Professor of Mathematics in the Mohammadan Anglo Oriental College, which was transformed in the year 1920 into the Aligarh Muslim University, was Babu Jadav Chandra Chakaravarti, a person well known in the mathematical world of India, who is still remembered with affection by his pupils. On his retirement Babu Chakaravarti was succeeded by Doctor (now Sir) Ziauddin Ahmad, who has ever since maintained a close connection with our Department of Mathematics and in spite of his multifarious activities, still lectures to the students of the Department in the capacity of Honorary Professor. Amongst those who have served the Department at various stages in the past, the names of Dr. O. H. Malik, Dr. Vijyaraghavan, Dr. Andre Weil and Mr. Kosambi may especially be mentioned.

I should like to avail myself of this opportunity, to draw your attention to the need of evaluating the contributions made to the advancement of Mathematics by the Hindu and Muslim Mathematicians.

Important contributions to the progress of the science—in the domain of pure as well as applied mathematics—were made by the Islamic world, particularly under the early Abbaside Caliphs, and the achievements of the Muslim Mathematicians present a vast and attractive field of work for the researcher. The exploration of this field, however, is no easy task and requires close and prolonged co-ordination between a body of Mathematicians and Arabicists.

The method of teaching the subject—particularly in our schools—is also one which deserves some attention from a Conference, comprising in its ranks Mathematicians and educationists. It is a regrettable fact that our system of instruction fails to inspire in a great majority of the pupils that love for the science of Mathematics which is the only way of promoting research at a higher

stage. The subject is taught from practically the lowest classes right up to the highest and a great deal of time and attention is devoted to it, but the result is not in keeping with the effort put forth. The question is one which needs examination and we shall have to revise and improve our system of instruction before we can hope to obtain results in consonance with our efforts and impart mental discipline which is the main justification for retaining the subject in the school curriculum.

In conclusion, I extend to you once more a most cordial welcome and hope that under the able guidance of an eminent Mathematician like Dr. Vaidyanathswami your deliberations will yield results of great value to the country and to the world at large.

Opening Speech

BY

LT.-COL. DR. SIR ZIAUDDIN AHMAD, *Vice-Chancellor, Muslim University*
(27th December 1941)

MR. PRESIDENT AND THE MEMBERS OF THE MATHEMATICAL CONFERENCE,

I first thank the members of the Mathematical Conference for their kindness in accepting the invitation of this University for holding their 12th session at Aligarh and I welcome the delegates who have made it convenient to respond to the invitation.

Our invitation to the Mathematical Conference was initiated by the late Sir Shah Muhammad Sulaiman whose loss on the present occasion we all deplore. In him the country lost a great jurist, a devoted scientist and an expert educationist. His researches and the incomplete work which he left may come up for discussion before this Conference. I am sure that I express the feelings of all the members when I say that we all miss at this Conference his personality at whose initiation we have assembled here today.

The researches in Mathematics are comparatively more difficult than those in other branches of science. In physical sciences, negative results have some value, but in Mathematics the negative results have no place. In every science the application of general principles to particular cases and their verification by using different sets of substances has some research value.

The chief difficulty in the Mathematical research is the absence of a well-equipped library supplied with current and back numbers of important periodicals. Some libraries have made individual attempts to collect books dealing with certain aspects of Mathematics, but no library can be said to be complete. It was suggested sometime ago that the Government of India should build up a well-equipped library either at Delhi or in Calcutta and that it should lend out books and periodicals to the University and College libraries. Every student of Mathematics who comes back after study in a European

University has certain problems on which he would like to continue his research, but due to the absence of opportunities he is unable to continue his research.

On account of the absence of contact between the teachers of different Universities and the want of up-to-date literature, research workers are very much handicapped in this country. This difficulty is more noticeable in problems affecting more than one science like the constitution of an atom. It is primarily a chemical problem but most of the researches are based on physical experiments and which are subsequently deduced by mathematical reasoning.

One would like to expect that the laws of nature governing the movements of bodies whether in a solar system or elsewhere must be the same. The use of spectroscopic measurement in recent years has attained a degree of accuracy which older methods did not possess. The minute measurement has revealed that Newton's law of gravitation which was universally adopted on account of its simplicity has failed to explain the new phenomena. Many physicists and Mathematicians have long been contemplating that Newton's law of gravitation was the first approximation of some other law which perhaps could not be expressed in terms of the functions already known to us. Einstein was the first who modified the law hy introducing the fourth dimension. Whatever the merits of his discoveries may be, he has left a permanent contribution not only in Mathematics but in Philosophy that space is not absolute. The researches of Michelson, Morley, Fitzgerald and others have proved that in all measurements, the motion of the observer should also be considered and they were led to the conclusion that the notions of time and space are inter-dependent. Einstein succeeded in explaining (1) the advance of the perihelion of mercury which is 43' per century, (2) spectral shift of light, and (3) deflection of light which Newtonion law had failed to explain.

The late Sir Shah Sulaiman started by adding the inverse of the fourth power to Newton's law of gravitation, and he took the co-efficient in the form in which it was given by Einstein. By his method he obtained a correction for the deflection of light which is fifty per cent more than the quantity arrived at by Einstein and even his figure is slightly short of the observation. Just before his death he was studying his own law, but could not complete his work.

Newton's law singularly fails in the motion of electrons, and especially when we deal with a velocity comparable to that of light. These are now explained by introducing the notion of quanta and by simplifying the differential equations by the use of highly complicated symbols. This has placed us in the same position in which celestial mechanics drifted when they attempted to explain heavenly phenomena by epicycles and hypocycles. The theory necessitated the addition of one more circle in order to explain a new phenomena discovered by more minute observations, so much so that King Alfonso once said that he would have created the universe in a simpler form. The universe is simple but our explanations are complicated. Lord Kelvin

once said that the discoveries in the physical sciences are forcing us to believe that we are on the verge of a great discovery which will explain the existing phenomena in a simpler manner.

I am glad that this Conference has invited a discussion on the teaching of Mathematics in Universities and Colleges. Our mathematical study is suffering for two main reasons:

Firstly, we have forced into the curriculum a number of subjects discovered in recent years and whose knowledge is essential at the present day without sufficient scrutiny and without modifying suitably the school syllabus. Thus the study of Mathematics in higher classes has a tendency to become more superficial, and this is enhanced by teaching Mathematics through languages which are not fully developed for scientific instruction. Secondly, we have not provided suitable courses of Mathematics for the students of Physics, and this is prejudicing the coordination of Physics and Mathematics.

Mr. President, the late Sir Shah Sulaiman planned the translation of the great work of El-Beruni, called *Kanoon Masoodi* written in 1038 whose publication had been continuously pressed ever since its publication was demanded by the French Academy of Science. El-Beruni knew Greek and Sanskrit and he studied Greek and Indian astronomy from original sources. Professor Sachau who was a great authority on El-Beruni and who published two of his important works, History of India and Chronology of Oriental people, once said that El-Beruni was the greatest intellect that ever lived on this globe. His book *Kanoon Masoodi* can be edited only by a ripe scholar of Arabic language thoroughly familiar with both ancient and modern astronomy, a rare combination.

I published a paper in 1905 on the third book of *Kanoon Masoodi* dealing with Trigonometry and drew attention to the following facts: regarding the Author:

(1) He is the first Mathematician who wrote an independent book on Trigonometry.

(2) He is the only Arab astronomer who took unity instead of 60 for the radius of the circle, and he defined six trigonometrical ratios by the lengths of straight lines. By this definition the geometrical properties of triangles can be interpreted in trigonometrical identities.

(3) He calculated the sine tables for every increase of 1° correct to seven places of decimals and tangent tables correct to five places of decimals.

(4) He calculated the value of sine 1° or π correct to fourteen places of decimals.

(5) Since the trisection of an angle is necessary for the calculation of sine 1° he gave 12 mechanical methods for trisecting by trial a given angle, which he naturally knew was impossible by theoretical geometry of lines and circles.

(6) He proved the six formulae of spherical right angled-triangles and also gave for the first time the proof of the formulae $\frac{\text{Sin A}}{a} = \frac{\text{Sin B}}{b} = \frac{\text{Sin C}}{c}$.

(7) He gave methods for finding the values of sine and tangent for intermediary angles. He employed the principle of proportionate parts for rough calculation, but for more accurate calculations he used second differences and he gave geometrical proof of the interpolation formula known after the name of Newton.

(8) He first introduced the notion of 'function' and generalised his interpolation formula to any function calculated for regular increase or decrease in the values of the variable.

(9) He attempted to calculate the radius of the earth hy finding the longitude of the place at which the peak of the mountain *Hindukush* was just visible at the horizon. He gave in the same book his own observations and calculated the value of the radius. El-Beruni was not familiar with atmospheric refraction and his results were not accurate. He realised the weakness of his results and throughout his book he accepted the length of 1° of the earth obtained by *Habash* and *Elfraghan* by actual measurements.

The third book of *Kanoon Masoodi* has now been published and we know the contributions of El-Beruni to Trigonometry. One chapter of his book dealing with Geography has just been published by the Archaeological Department of the Government of India. Sir Charles Elliot wrote a memorandum on this subject which is now in the British Museum. El-Beruni's researches in Celestial Mechanics and Physical astronomy are not fully known. We have glimpses of his contributions to other branches of astronomy, a few of which may be described here.

(1) He was familiar with Telescope which is supposed to have been invented by Galileo.

(2) He has given the right ascension and declination of 1029 stars by his own observations.

(3) In Chapter II Book IV he described the nebulae in detail and he described them as cluster of stars.

(4) In Book IV he mentioned that several stars have shifted from the position given by ancient astronomers and mentioned particularly the star in the heart of Leo which he said was not in its old position.

El-Beruni begins the discussion on every new topic by giving an account of the researches of his predecessors, Greeks, Arabs and Indians on the subject, before expounding his own views. Hence the correct history of astronomy cannot be apprehended until *Kanoon Masoodi* of El-Beruni is published in modern notation.

A notable example is the discovery of the secular acceleration of the moon's mean motion which Halley discovered by comparing the tables of Ibne Yunus and Ptolemy and which he fixed at 10″ per century. It was left to Adam and De Launy to examine the cause of this secular variation and to calculate more accurately its value, which according to them was 6″·176 per century. Newcomb observed that Arabian observations are far more reliable than those of Ptolemy and found the variation to be 7″ by comparing modern observations with the observations of Ibne Yunus. The publication of *Kanoon Masoodi* will throw a great light on the influence of Indian and Greek astronomy on the researches of the Arabs.

Mr. President, I request you to commence the formal proceedings of the Conference.

SECRETARY'S REPORT

BY

DR. RAM BEHARI, *Secretary of the Society*

1. On behalf of the Managing Committee of the Indian Mathematical Society, it is my pleasant duty to extend to you all, a hearty welcome to this Conference.

We are specially grateful to you, Sir, for the trouble you have taken, in spite of your multifarious duties to come to inaugurate our deliberations.

2. It is with deep sorrow that we have to record the heavy loss which the Society has sustained in the deaths of :—

(i) Prof. M. T. Naraniengar, formerly Professor of Mathematics at the Central College, Bangalore, who died at Bangalore on the 9th October, 1940. Professor Naraniengar was the Foundation Editor of the Journal of the Indian Mathematical Society and laboured unceasingly and almost single-handed for twenty years to make it a worthy embodiment of the creative activity of India in the field of higher Mathematics. He was President of the Society during the years 1930-32 and was universally loved and respected for his high sense of duty and the simplicity and saintliness of his character.

(ii) Sir Shah Mohammad Sulaiman, formerly Vice-Chancellor of the Aligarh University and Judge of the Federal Court, who died at Delhi on the 12th March 1941. Sir Shah Sulaiman was an active member of tbe Managing Committee of our Society, and about a fortnight before his death, as Vice-Chancellor of this University, he had invited our Society to hold its Twelfth Conference here. He was an exceptionally gifted lawyer, a keen Mathematician and a great patron of learning, and was admired and loved by all fellow-workers in the field of Science or Law.

3. Our Society was founded by the late V. Ramaswamy Aiyar and a group of enthusiastic associates in 1907 to promote the advancement of

Mathematical Study and research in India, and since then it has been growing steadily in strength and usefulness under the wise guidance of its successive presidents.

During the time that has elapsed since we last met at Hyderabad-Deccan in December 1939, the Society has fairly maintained its strength. 19 new members have been enlisted including one life member.

4. Our Society has three main activities viz., organizing the biennial Conferences, the publications of two Journals and the maintenance of a library of advanced books and journals which are circulated to members who ask for them.

The first conference of the Society was held at Madras in December 1916. Since then we have had conferences at Bombay, Lahore, Poona, Bangalore, Nagpur, Trivandrum, Delhi, Lucknow and Hyderabad (Deccan). We are now holding our Twelfth Conference in this centre of learning under the auspices of the Muslim University.

It is very encouraging to note that in spite of anxious times through which our country is passing on account of war which has spread to our very doors, our members and well-wishers have assembled here in large numbers to make this conference a success by their presence and mathematical contributions. More than 50 papers have been communicated to this conference. We have also been able to arrange three symposia on modern research topics, three popular lectures on interesting subjects, and a discussion on the courses and methods of teaching mathematics.

At the business meeting of the last conference at Hyderabad-Deccan, a committee consisting of Dr. R. Vaidyanathaswami, Dr. A. N. Singh and Professor N. R. Sen was appointed to consider means for securing a closer co-operation between the Benares, Calcutta, and the Indian Mathematical Societies. It is gratifying to note that the Benares Mathematical Society will be holding its annual meeting during our conference session tomorrow.

At the suggestion of the Hyderabad conference, a committee was appointed to examine the place of Mathematics in Higher Competitive Examinations. A memorandum was sent to the Secretary of the Federal Public Service Commission and a very satisfactory reply was received from him.

5. The Society continues to publish two quarterlies. *The Journal of the Indian Mathematical Society* is now managed by a Board of Editors with Dr. Vaidyanathaswami as Chairman and Dr. C. N. Srinivasiengar as the Secretary. It publishes only original papers and caters for the needs of workers in the field of advanced mathematics. The other Journal, under the title, '*The Mathematics Student*' is edited by Dr. A. Narasinga Rao of the Annamalai University and is dedicated to the service of students and teachers of Mathematics in India. It serves as a link between higher mathematics

and the sphere of collegiate mathematics, and also publishes 'News items' with the object of keeping the readers in touch with one another and with the leading events in the mathematical world in India and abroad.

During the period 1940-41, 65 papers were received for publication in the *Journal.* It is gratifying that the quality and quantity of mathematical research work submitted for publication is fast improving. Owing to the large number of papers awaiting publication, the size of the Journal was increased from about 36 pages to 48 pages per issue, with effect from December 1940; the get-up and type were also improved, but the new bolder types employed have more or less neutralised the increase in the number of pages and the question of increasing the size of the Journal requires consideration again.

The large number of papers awaiting publication is partly due to the War, since a number of papers which would have gone out to foreign periodicals are now being published in our Journal, but there is every reason to hope that the flow of good papers to our Journal will be maintained at the present level, even after the War.

The *Mathematics Student* continues to serve the student world by critical treatment of collegiate mathematics and by drawing attention to interesting topics and research problems. Undergraduate students and research scholars have been contributing regularly to its columns though the response has not been as plentiful as was hoped. Since it serves as a training ground for young mathematicians in acquiring the art of presentation of their investigations clearly and without the central idea becoming obscured by details, many of the papers have had to be revised (and sometimes rerevised) before publication. The *Student* has appeared without a break four times a year, though somewhat after its due date.

Consequent on the war, the cost of printing is continuously going up. On the other hand, in spite of persistent efforts, it has not been possible to increase our income proportionately. Authors of papers are therefore requested to make them as brief as possible; they are also requested to return the proof-sheets of their papers promptly to avoid unnecessary delay in their publication.

The Society is grateful to all the referees, who have most willingly and at great personal sacrifice assisted the Society with their expert advice on papers sent to them for opinion.

6. The Library of the Society continues to be under the charge of Professor R. P. Shintre and Professor D. D. Kosambi, of the Fergusson College, Poona. Owing to the international situation the number of available current Mathematical periodicals has diminished very considerably. The number of Journals received in exchange has come down from 69 to 36. No foreign journal published outside the United States of America and the British Empire is now being received. Even those still received, arrive irregularly and at much

longer intervals than before. The Librarian wishes to get all the old issues bound. Members are, therefore, requested to return all old issues that they can locate, and are warned that the circulation system may have to be suspended without further notice.

7. Regarding the rates of subscription for members the general body suggested to the Managing Committee at the last conference an examination of the life composition fees and the consideration in particular, of the proposal for a uniform rate of Rs. 10/- per annum for all members. The managing committee gave its best consideration to this proposal and resolved that the existing rates be not changed as the adoption of the uniform rate would adversely affect the finances of the Society.

The Managing Committee hopes that with the willing co-operation of our members, the financial position will improve by enlisting new members and institutional subscribers for the journals, and that more Universities, Provincial Governments and Ruling Princes will come to our aid by making handsome annual grants.

We are very glad to announce that Dr. A. Narasinga Rao of the Annamalai University has offered a prize for encouraging research in Mathematics to be awarded at our conferences. The conditions of award are being considered by the Managing Committee.

8. In conclusion, I take this opportunity of expressing our thanks to the Universities that have rendered financial help in the past by giving annual grants-in-aid, to the various local Governments, Indian States, Universities and other institutions for the facilities they have granted to their officers to enable them to attend this Conference.

I have also great pleasure in expressing, on behalf of the Managing Committee, our thanks to the Vice-Chancellor of the Aligarh University for giving us this opportunity of assembling here for this session.

Our Special thanks are due to Mr. S. M. Kerawala, his band of volunteers, and to the members of the Local Reception Committee for their warm hospitality and excellent arrangements for the comfort of the guests.

GLEANINGS

From childhood to old age Gauss always used the empty pages and the inside of the book covers as writing paper, not only for the entry of formulas often used and of occasional minor calculations, but frequently also for abstruse researches.

From G. W. DUNNINGTON'S
CARL FRIEDRICH GAUSS.

PRESIDENTIAL ADDRESS

BY

DR. R. VAIDYANATHASWAMY, *Madras University.*

Mr. Vice-Chancellor, Ladies and Gentlemen:

It is with great pleasure that I acknowledge the warmth of the welcome which has been accorded to this Mathematical Conference. When I look back on the previous conferences of the Indian Mathematical Society, it seems to me that the present conference will prove to be memorable in many ways. In the first place, in contrast to the peaceful atmosphere of our Mathematical deliberations in the past, we are meeting for the first time under the threat of impending war-conditions. The association between the tense exigency of the moment and the calm detachment of scientific thought is not perhaps so unnatural as it may appear at first sight; our temporal consciousness moves always between the two poles of the moment and the eternal, but these two poles even in their opposition seem to be linked in a mysterious kinship; for how else can we explain the fact that often in moments of intense stress, our minds get a sudden access to sensibility and becoming as it were, released from a prison, rise to a perception of universal things? We may recall that Archimedes, who with Newton and Gauss is reckoned among the three greatest Mathematicians of the world was in a state of concentration on a mathematical problem when the invaders of his city broke in upon him. Again, this conference is the first held by the Indian Mathematical Society in collaboration with a sister Mathematical Society; and this co-operation for a common cause is a happy augury for the future advancement of Mathematical research in this country. And lastly, this is the second time in succession that our Conference has been welcomed by a progressive University devoted to Muslim Culture—a culture that has made a respectable contribution to the early development of our Science.

The fact that in the past our Conferences have been enthusiastically received and welcomed by various Universities shows that, though Indian Universities are mainly engaged in the dissemination of knowledge, they are not blind to the fact that the primary function and justification of the University is research or the creation of new knowledge. It appears to me that the Universities in India are in a stage of transition at the present day. On the one hand, they are fettered by their past record of a mechanical reception of Western Knowledge and its mechanical transmission. Now, when the ideal of vernacularisation of all teaching is in the air, they are showing signs of discomfort and are beginning to be seized by stirrings towards a more creative and a fuller intellectual life. While we look forward to a post-war India with a revised political status, it will not I think be any undue optimism, to hope that Higher Education in this country will come to its own, and the Indian Universities will begin to function as the centres of higher knowledge and research and intellectual leadership in the arts and sciences.

It will certainly prepare the way for this inevitable transformation, if we realise that college instruction cannot be separated from research except to its detriment and degradation. The syllabus in Mathematics for a degree course, for instance, consists of a body of doctrines, together with the training for manipulating them in solving certain types of questions; these doctrines are advanced in text-books and by teachers, as if they had no background and no connection with any large human interests; as if they had always existed in their own right, because they were absolutely true and final—with that species of brutish finality which popular opinion finds it easier to attribute to Mathematics, than to sciences which are partly or wholly experimental. But this is a totally distorted view of the matter. The doctrines in our Mathematical syllabus represents the net gains of some two milleniums of intellectual effort of the human race; each particular idea or relationship was first manifested to an individual human intelligence after a long travail of seeking and the whole body of doctrine called Mathematics has been gathered and handed down as a precious relic, because of its inestimable value and significance for those ideals and for that fulfilment, for which the human race is striving through the ages. If this outlook towards Mathematical truth cannot be instinctively communicated by the teacher to his pupils, then what the teacher performs is not the teaching of Mathematics but its mechanisation and vulgarisation. And as a rule, the teacher will not be able to communicate this kind of feeling for Mathematics, unless he has himself gone through the great experience of Mathematical discovery. To the Seeker after Truth, this cardinal experience of discovery is a turning point, and may well be termed a baptismal experience, as it marks his birth into a world of deeper reality—the world of ideas. The experience of discovery is roughly the glimpsing of a new idea, by which the past knowledge is re-ordered and re-oriented, so as to exhibit hidden relationships and new and unsuspected values, so that there results a thorough transformation of outlook. A prosaic description of this kind does not really convey the sense and momentousness of this cardinal experience of the enlargement of human consciousness; for that, one must hark back to the Vedic and Puranic Symbolism. The Ushas of the Rig Veda, the Deity of the Dawn is nothing else but this experience, and the glamour of this Deity, and the number of the hymns addressed to her and their fervid passion, show how deeply the Vedic Rishis had realised and experienced the recurrent enlargements of the aspiring evolving human consciousness. Similarly the glamourous figure of Lakshmi rising from the ocean of milk churned by the Devas and Asuras is the pre-eminent symbol of something absolutely new and original, something fresh from the primordial mint of creation, rising in the human consciousness and miraculously transforming the whole outlook and revealing a new world of values. It is only that person who has experienced the great cosmic mystery of the rising of the original Lakshmi from the churned Ocean of Milk that can penetrate behind the Symbolism of the cheap vulgarisations of the figure of Lakshmi, which can be bought in the bazaars; in a similar manner, it is only the teachers who have gone through the baptismal experience of Mathematical discovery, that can convey to his pupils the inner

significance of the text-book Mathematical doctrines, relive with them the original glamour of their first discovery, their first revelation to a thrilled human mind, and communicate a feeling of their place and value in the journey of the human race, the Epic of Civilisation.

The Indian student or Indian Teacher of the present day wishing to undertake Mathematical research, is confronted with the vast Mathematical material which is the legacy of the 19th century. The 19th century has been well called the Golden Age of Mathematics, its achievements belittling that of all previous centuries taken together. The developments of this century in "Analysis" and "Function-Theory", in Algebra and Geometry, in Arithmetic and in Invariant-Theory, in Mechanics, have so changed the face of Mathematics as to alter it beyond recognition and have won for it a far-flung dominion. Indeed some of the things which are a result of the work of this century, and which we teach young pupils of the College classes belong to a conceptual level that would baffle the understanding of the mighty intellects of antiquity or of the Middle Ages. The basic ideas of the 19th century work do form or should form part of the course in our Honours and post-graduate classes. The Mathematical aspirant who wishes to push on to research, should of course discipline himself thoroughly in the methods of great masters of the 19th century; this would not be difficult to accomplish in any particular topic, though the extent and refractory character of the material would make it difficult to do in several topics at once. The worker would then be naturally led on to extend the scope of these methods or to complete the solution of some problem by taking it up at the point where it had been left off. This would be the usual way which workers might be expected to take and do actually take both here and in other countries. But is this the right way? I think not. Let me explain why. Some young Indians inspired and fascinated by Euclid's Elements, are enticed to further research and do succeed in finding ingenious proofs of difficult riders or in constructing elegant theories within the ambit of Euclid's method and view-point. Others, of a more skeptical or valorous disposition attempt a proof of the parallel postulate by the classical methods of Euclid; these are by no means cranks or irrational persons; for if you limit yourself to the view point of Euclid, it is not at all certain that the parallel postulate could not be proved. Euclid, of course, guessed that it could not be, but it was only a guess, and he might have well been wrong. It was only in later times and from a wider point of view that it became possible to prove that the parallel postulate cannot be proved. My point is that in neither of these cases is there any really creative work, because there is no advance beyond the point of view of Euclid — — the point of view of two milleniums ago. On the other hand what the true nature of creative work in geometry should be is well demonstrated by the actual historical course of development. The transformation of Euclidean Geometry into Geometry as we think of it today was effected by the impact of some half-a-dozen new ideas in succession on the material of Euclidean Geometry. Each of these impacts irradiated the material and raised it to a higher plane of value and significance. The Treatise on the Conic

Sections of the Greek Geometer Appolonius was *not* one of these impacts, because though an extension and addition to Euclid's work it was not a *creative* addition in the true sense, it belonged to the same value-level and the same significance-level as Euclid's elements. These impacts may be easily enumerated; the first was Descartes' discovery of Algebraic Geometry, the second was the principle of continuity associated therewith, which brought in imaginary elements into geometry and the concept of n-dimensional geometry. The next was the discovery of non-Euclidean Geometry with the conviction of its *reality*. The next was the discovery of projective geometry with the importing of infinite elements. The next was the concept of group and invariant culminating in Klein's grand generalisation of geometry as the invariant theory of a transformation group. The last was Gauss' discovery of differential geometry and Riemann's concept of curvature of a manifold. Each of these new ideas illuminates the subject matter in such a way that it might well be called a "DAWN" in accordance with the Vedic imagery; and this name is all the more appropriate because the illumination in each case has come in response to a seeking for light in order to clear up some felt darkness in the subject. Thus in brief Euclid's conception of geometry and ours are separated by some half a dozen dawns.

This being the nature of creative research, the present day Mathematical aspirant should seriously ask himself whether in dealing with the rich legacy of the 19th century, he is really bound to limit himself to its view point and method, whether there have not happened subsequent dawns, dawns of this century, the illumination of which has yet to play on the older material transforming it and raising it to its own level of value and significance. If there have been such dawns, the path of creative research lies most definitely in the fulfilment of the Time-Spirit by assimilating the older material to the newer light.

We have today arrived at a deeper insight into the nature of Mathematical thought than was ever possible before. As the culmination of the researches in the analysis of mathematical reasoning initiated in the last century, we realise today that Mathematics should be conceived as 'the class of all deductive systems' or as 'the analysis of structure'. This is no doubt a highly abstract view, but abstraction where Mathematics is concerned, is not a flight from the concrete, but a temporary stepping back from the subject matter, in order to get it into clearer perspective and return upon it with greater power and insight. Looking back on the past, we may recognise that this intuition of the nature of Mathematics has been always present from the earliest times as far back as Euclid, though in a germinal form without rising to self-awareness.

Another curious feature of this modern view of Mathematics is the laying bare of the peculiar relation which holds between Mathematics and Logic—a relation, not of one-sided dependence as was thought, but of two competitors in a race each of whom tries to outstrip the other. For, it turns out that logic—or rather, any particular level or system of logic—has a structure of its own and

can be formalised and presented as a deductive system, in which the processes and reasonings depending on the actual 'content' as opposed to the 'form' of the system belong to a deeper unformalised level of 'meta-logic.' This must necessarily turn out to be the case with a human mentality capable of deeper and ever deeper levels of reflection on its own processes.

The 'Structures' or 'Deductive Systems' of Mathematics may be of a very general or even of an arbitrary character. But in practice, the purposive character of Mathematical science brings in some limitations. Historically Mathematics had the task of elucidating the problems of practical life. Its central and distinctive contribution in this regard is the concept of the 'real number,' by which the sense world can be 'measured,' and its manifold relations explained. This concept runs like a spinal cord through the whole of Mathematics, furnishing a solid support to its abstract structures and a centre of reference to its boundless peregrinations Indeed, when any general Mathematical theory or result has to be brought home by an illustration or to be used or applied, we have to bring in the real number directly or indirectly. It will not be unnatural then to except the typical Mathematical Structures to have a close bearing on the real number concept, to hover round it, as it were, or even to have been implicitly involved in its development.

The concept of the 'real number' has developed through four stages, the cardinal number, the natural number, the rational (or fractional number), and the real number, and we do find Mathematical Structures typical of each of these stages of its evolution. The 'Cardinal number' is the signless integer used in counting, and the addition of these cardinal numbers must be explained in terms of the union of sets without common elements. Thus the notion of cardinal number must be logically explained in terms of set-algebra. But the notion of set has a definite content, and is therefore 'concrete' and not 'abstract' according to the standards set by Mathematics. To get at the formal characteristic of set-algebra, we observe that both set-union and set-intersection can be explained in terms of the relation of set-inclusion. Therefore the abstract Mathematical Structure revealed in set-algebra and in the notion of cardinal number is the 'partially ordered set,' namely a set of elements constituting the field of a binary-reflexive transitive relation, formally analogous to set-inclusion. It is not surprising that the structural idea of the 'partially ordered set' includes logic also among its applications, in view of its close connection with the fundamental logical notion of 'set.'

The cardinal numbers are not closed for subtraction, and the impulse which conceives the 'natural numbers' or the set of positive and negative integers, is clearly the impulse of 'completing the group.' Thus the Structural idea which is manifested at the second stage of evolution is the idea of 'group.' The passage to the rational number is similarly effected by completing the multiplicative group of the natural numbers other than zero; this brings forth the structural idea of 'field' which is or rather ought to be, the subject-matter of Algebra. In the final stage, the notion of real number is arrived at from

the rational number by a *coup d'etat* accepting an infinite process. According to Dedekind, the motivation in this process is to arrive at an analysis and a logical explanation of the geometrical intuition of 'Continuity.' The Mathematical discipline which analyses and investigates the structure of the notion of continuity is called 'Topology.' Accordingly the structural moment revealed in the passage from the rational to the real number must be called the 'topological moment.' Thus corresponding to the four stages of evolution of the real number, we have four principal structure-types, the 'partially ordered set' and 'topology' at the extremes, and the intervening structural levels of group' and 'field' which are usually included under 'Algebra.' These are of course 'elementary' structure forms, and may be found mixed up together in the background of any actual piece of mathematical reasoning. In order to study these structures effectively, Mathematics would also have to deal with generalised forms of a looser structure, as for instance 'Semi-group' or 'ring.' Also, an infinite variety of secondary or derived structure-types may arise, based on these; as for instance in Geometry, where we have to deal with the structure of sets of linear equations in a field. Thus, in fine, we have to ascribe a central position in Mathematics, to the four structural moments revealed in the evolution of the real number, without prejudice to the fact that Mathematics is concerned with deductive theories based on any system of postulates whatever.

Every discovery of new and far-reaching ideas must be followed up by a period in which their scope and applicability to the existing material is surveyed and worked out. Such work is necessary not only for an increased insight into the existing body of knowledge, but also for giving content to the new ideas and fixing their bearings. It is work in this direction that is called for at the present moment, and I feel strongly that Indian Mathematicians and workers should orient themselves accordingly, so as not to be behindhand in doing their share.

GLEANING

Few people realize how much the geometric ideas of symmetry, congruence, equality, and similarity influence their lives. The autombile in which we ride has many applications of these principles. The stamping dies press into form sections of the body and many of its fittings. These are produced as congruent parts by simple repetition. If this were not true, a car would cost several times its present price. The difference in the cost of tailor-made and factory-made suits is the result of the application of congruence and similarity. Many congruent portions are made at a single cutting by machinery. Sizes 30 to 49 exemplifies similarity, the suit as a whole represents symmetry.

A. B. Miller in *The Mathematics Teacher* Dec. 1941.

ABSTRACTS OF PAPERS

P. Kesava Menon, Annamalai University.

A Test for Groups of Primes.

In a paper entitled 'A test for groups of primes'* Dr. S. S. Pillai has obtained the following results:—

(i) the necessary and sufficient condition that both p and $p+r$ be primes is

$$(p-1)!+1+\frac{p(r!-1)}{(r\cdot r!)}\equiv 0 \bmod p(p+r),$$

and

(ii) the necessary and sufficient condition that the three numbers p, $p+r$, $p+s$ be primes is

$$(p-1)!\ +1+\frac{p(r!-1)}{r\cdot r!}+p(p+r)\left\{\left(\frac{r!-1}{r\cdot r!}-\frac{s!-1}{s\cdot s!}\right)\Big/(s-r)\right\}$$
$$\equiv 0 \bmod p\ (p+r)\ (p+s).$$

He further states that these results may be extended to groups of larger number of primes but that he has not been able to reduce the conditions for the general case to any simple form. In this paper I prove the following general result:

The necessary and sufficient condition that

$$p+r_0,\ p+r_1,\ .\ .\ .\ .\ ,\ p+r_n\ \ (r_0=0)$$

be primes is

$$(p-1)!\ +1+\sum_{i=0}^{n-1}(-1)^{i+1}\,a_{i+1}\,(p+r_0)(p+r_1)\ .\ .\ .\ (p+r_i)\equiv 0$$
$$\bmod p(p+r_1)\ .\ .\ .\ (p+r_n),$$

where

$$a_i=\sum_{j=0}^{i}\frac{1}{r_j!\,\phi_i'(r_j)},\qquad \phi_i(x)=(x-r_0)(x-r_1)\ .\ .\ .\ (x-r_i).$$

D. R. Kaprekar, Devlali.

On Superwonderful Demlo Numbers

Numbers of the form

$$W_p=123456789_{(p)}87654321$$

where $9_{(p)}$ stands for the digit 9 repeated p times are considered in

* *Journal of the Indian Mathematical Society* Vol. XVII (1927).

this paper. With the help of a table of recurring periods of M/81 for different values of M from 1 to 80, rules are given for writing down the product of a W_p with any integer with little effort. Such products are sometimes very striking. Thus

$$W_8 \times 1377 = 1699999998283000000017$$

S. S. Pillai, Travancore University.

1. *On the number of numbers which contain a fixed number of factors*

Let (1) $\sigma_\nu(x)$ denote the number of numbers $\leqslant x$, which are composed of ν prime factors, where each factor is counted according to its multiplicity; (2) $\pi_\nu(x)$ denote the number of such square-free numbers, and (3) $\rho_\nu(x)$ denote the number of numbers $\leqslant x$, which are composed of ν *different* prime factors.

Then when ν is fixed, Landau has proved in '*Primzahlen*' (pp. 202—213), that

$$\sigma_\nu(x) \sim \pi_\nu(x) \sim \rho_\nu(x) \sim \frac{x}{\log x} \cdot \frac{(\log \log x)^{\nu-1}}{(\nu-1)!}.$$

When ν is a function of x, nothing is known.

In this paper, I prove that the above asymptotic formulae hold good when $\nu = O(\log \log x)$; further I show that if x is sufficiently large,

(i) when $\nu < k \log \log x$, there is a constant $H = H(k)$ such that

$$\pi_\nu(x) > H \cdot \frac{x}{\log x} \cdot \frac{(\log \log x)^{\nu-1}}{(\nu-1)!};$$

(ii) when $\nu < 1{\cdot}015 (\log \log x - 1{\cdot}7)$,

$$\sigma_\nu(x) > \frac{5}{27} \cdot \frac{x}{\log x} \cdot \frac{(\log \log x)^{\nu-1}}{\nu-1!},$$

and

(iii) when $\nu < 1{\cdot}00035 (\log \log x - 3{\cdot}2)$,

$$\pi_\nu(x) > \frac{1}{32} \cdot \frac{x}{\log x} \cdot \frac{(\log \log x)^{\nu-1}}{(\nu-1)!}.$$

In the body of the paper other results will be found.

There is reason to guess that the asymptotic formulae hold good for $\sigma_\nu(x)$ where $\nu \sim \log \log x$. It is intensely interesting and at the same time terribly difficult to find out the critical value of ν when

the asymptotic formula breaks down. Even a guess seems to be impossible in the present state of knowledge.

2. *On a lattice point Problem*

Let $f(x)=x-[x]-\frac{1}{2}$ when x is not an integer and $f(x)=0$ when x is an integer; and let $F(\theta, n)=\sum_{r=1}^{n} f(r\theta)$. The paper discusses the properties of $F(\theta, n)$.

Hansraj Gupta, Govt. College, Hoshiarpur

On Numbers of the form $4^a(8b+7)$

All numbers which cannot be expressed as the sums of less than four squares are of the form $4^a(8b+7)$ where a and b are non-negative integers. If $K(n)$ denotes the number of such integers not exceeding n, this paper is concerned with an expression for $K(n)$ and a method of calculating $K(n)$. The results are an improvement on those of Mr. S. C. Chakrabarti in *Bull. Calcutta Math. Soc.* 32 (1940).

The paper also deals with the number of solutions of the equations

$$n-6K(n)=m, \quad 0\leqslant n<4^s$$

and

$$n-6K(n)=m, \quad 4^s\leqslant n<4^{s+1}, \quad s\geqslant 1.$$

M. Ziaud Din, Lahore.

On partitions and divisors of a number

Symmetric functions and determinants have been applied to derive certain formulae in partitions and divisors of an integer.

F. C. Auluck, Lahore.

On the maximum value of the number of partitions of n into k parts

M. Venkatarama Iyer and M. Bhimasena Rao, Bangalore.

On types of solutions of $x^3+y^3+z^3=1$ in integers

K. Sambasiva Rao, Andhra University, Waltair

On the function $\gamma(k)$ of E. M. Wright

We call a solution in positive integers of

$$x_1^k + x_2^k + \cdots + x_m^k = y_1^k + \cdots + y_n^k \tag{1}$$

non-trivial if

(A) no x is equal to any y,

and (B) $(x_1, \cdots, x_m, y_1, \cdots y_n) = 1$.

E. M. Wright * denotes by $\gamma(k)$ the least value of n such that (1) has infinitely many non-trivial solutions with $m < n$ and proves that
$$\gamma(k) = O\{(1{\cdot}307)^k\}.$$

S. Chowla† defines $\gamma(k)$ as above but without the restriction (A) and proves that
$$\gamma(k) < \tfrac{1}{2}(k^2 + k) + 1.$$

In this paper the author considers the functions $\gamma(k)$ of Wright, and proves that
$$\overline{\lim_{k \to \infty}} \frac{\gamma(k)}{k \log k} < 2.$$

K. Chandrasekharan, Madras University.

On the (J_μ, λ) summation of Series

In an unpublished memoir, Dr. S. Minakshisundaram has made use of a new method of summation which is defined as follows :—

A Series $\sum a_n$ is said to be summable (J_μ^k, λ) to the sum S, if

(i) $\phi_\mu^k(t) = \sum_{n=0}^{\infty} a_n \, \alpha_\mu^k(\lambda_n t)$ is convergent for real $t > 0$,

(ii) $\phi_\mu^k(t) \to S$ as $t \to 0$,

where k is a positive integer, (λ_n) is an increasing divergent sequence of positive numbers, and $\alpha_\mu(x) = 2^\mu \Gamma(\mu+1) \dfrac{J_\mu(x)}{x^\mu}$, $J_\mu(x)$ denoting the Bessel function of order μ.

* E. M. Wright: "On Sums of kth powers," J. L. M. S., Vol. 10 (1935), pp. 94-99.

† S. Chowla: "A Theorem on Sums of Powers with applications to the additive theory of Numbers." *Proc. Ind. Acad. Sci.* Vol. I (1936) pp. 701-6.

The object of this paper is to prove the following results expressing the relationship between the above process and Cesàro's or Riesz's process as the case may be :—

I. If $\sum a_n$ is summable (Cr), then it is summable (J_μ^k, n) for $\mu > \frac{r+1}{k} - \frac{1}{2}$

II. If (i) the series $\sum a_n \, a_\mu (\lambda_n t)$ converges for all $t>0$, and uniformly in every finite interval,

(ii) $\phi_\mu (t)$ is bounded as $t \to \infty$,

(iii) $\phi_\mu (t) \to s \neq \infty$ as $t \to 0$,

then $\sum a_n$ is summable $(R ; \lambda, k)$ for $k > \mu + \frac{1}{2}$.

V. S. Krishnan, Madras University.

Homomorphisms between distributive Lattices

Homomorphisms between lattices have not so far been considered in any detail. The purpose of my paper is to study general homomorphisms between distributive lattices—which preserve finite sums and products, and the special cases of these, called by me 'Regular,' which are determined solely by ideals in the first lattice. By establishing a regular homomorphism between a lattice L, in which every element has a product complement, and this set N of normal elements of L, it is possible to study the normal elements from an entirely new point of view.

P. D. Shukla, Lucknow.

On the differentiability of the integral.

The necessary and sufficient condition for the existence of F′(0), where

$$F(x) = \int_0^x \phi \, [\psi(x)] \, dx,$$

and where $\psi(x) \to \infty$ as $x \to 0$, and $\phi[\psi]$ is any finite and continuous function of ψ having a discontinuity of the second kind at $x=0$, has been investigated. Under certain restrictions on the function $\phi[\psi(x)]$, this problem has been studied earlier by the author (*Proc. Ind. Sc. Congress, Benares* (1941), Pt. III, pp. 6-7). In the present paper those restrictions have been removed and the necessary and sufficient condition for the existence of F′(0) has been found out in the most

general case of ϕ. An attempt has also been made in the present paper to show that the results about $F'(0)$ as found out in a special case of ϕ by G. Prasad are the same as those found by the author in the general case of ϕ.

S. M. Shah, Muslim University, Aligarh.

1. *On the lower order of an integral function.*

I define the lower exponent of convergence of zeros of $f(z)$ and prove two theorems.

THEOREM 1. *Given $\rho\,(0<\rho<\infty)$ there exists an integral function $f(z)$ of order ρ for which $\rho=\rho_1$ but for which $\lambda>\lambda_1$*

Here ρ_1 is the exponent of convergence of the zeros and λ the lower order of $f(z)$.

$$\lambda_1 = \varliminf_{r=\infty} \frac{\log n(r)}{\log r}$$

is the lower exponent of convergence of zeros of $f(z)$.

THEOREM 2. *If $f(z)$ be of order ρ, $0<\rho<1$ then $\lambda \leqslant \dfrac{\lambda_1}{\lambda_1+1-\rho}$*

2. *Note on a Theorem of Polya.*

Let $f(z)$ be an integral function of order ρ G. Polya proved that

$$\varliminf_{r\to\infty} \frac{n(r)}{\log M(r)} \leqslant \rho \tag{1}$$

where $M(r)$ and $n(r)$ have their usual meanings. Valiron showed that if $f(z)$ be of finite order ρ.

$$\varliminf_{r\to\infty} \frac{n(r)}{\log M(r)} \frac{\log r}{\log\log M(r)} \leqslant 1 \tag{2}$$

In this note I prove a better result, namely.

If $f(z)$ be an integral function, and if

$$\lambda_1 = \varliminf_{r\to\infty} \frac{\log n(r)}{\log r}$$

then

$$\varliminf_{r\to\infty} \frac{n(r)}{\log M(r)} \leqslant \lambda_1. \tag{3}$$

V. Ganapathy Iyer, Annamalai Unversity

A property of the maximum modulus of integral functions

Let E be a measurable set in $(0, \infty)$. Let $E(r)$ denote the measure of E in $(0, r)$. Then

$$\overline{\lim_{r\to\infty}} \frac{E(r)}{r} \text{ and } \underline{\lim_{r\to\infty}} \frac{E(r)}{r}$$

are called the upper and lower densities of E.

Now let $f(z)$ be an integral function and $M(r, f) = \max_{|z| < r} |f(z)|$. The order ρ is defined by

$$\overline{\lim_{r\to\infty}} \frac{\log \log M(r)}{\log r} = \rho$$

Let E be the set of points in $(0, \infty)$ such that

$$\lim \frac{\log \log M(r)}{\log r} = \rho$$

as $r \to \infty$ over E. I show that E has upper density one and that it is the best possible result in the sense that there are functions for which the lower density of E is zero. I prove a similar result for functions of order zero but of finite logarithmic order. I apply these results to study the precise order of composite integral functions.

R S. Varma, Lucknow University.

Some infinite series involving Humbert Functions

In this paper I have investigated some infinite series involving Humbert Functions. The product of two Humbert Functions is expressed in terms of generalised Hypergeometric functions. The problem of convergence is discussed with the help of the asymptotic behaviour of the function recently obtained by me in a paper published in the April 1941 issue of *Annals of Mathematics.*

P. C. Mital, Lucknow University.

Operational images of functions which are self-reciprocal in the Hankel Transform.

In this paper, I have given two theorems for investigating the operational image of a function which is self-reciprocal in the Hankel Transform.

Hari Shanker, Anglo-Arabic College, Delhi.

On Hankel Transforms of the product of two parabolic cylinder functions.

The object of this paper is to establish the relation

$$\int_0^\infty D_\mu(x)\, D_{-\mu-2n-1}(x)\, x^\nu\, J_\nu(ax)\, dx$$

$$= \frac{2^\mu\, \mu!\, \Gamma(2\nu+1)\, 2^{(n-3\nu-1)/2}\, a^{n-1}\, e^{\frac{1}{4}a^2}}{\Gamma(\nu+1)} \times$$

$$\sum_{r=0}^{\mu} \frac{\Gamma(n+r+\frac{1}{2})(-\sqrt{2}a)^r\, W_{-\frac{(n+\nu+r)}{2},\, \frac{n-\nu+r}{2}}(\tfrac{1}{2}a^2)}{(\mu-r)!\; r!\; \Gamma(2n+r+1)}$$

and prove the following Theorem —

If $\phi(x) = x^{\nu-\frac{1}{2}}\, D_\mu(x)\, D_{-\mu-2n-1}(x)$

and $\psi(x) = \dfrac{2^\mu\, \mu!\, \Gamma(2\nu+1)\, x^{n-\frac{1}{2}}\, e^{\frac{1}{4}x^2}}{\Gamma(\nu+1)\, 2^{\frac{3\nu-n+1}{2}}} \times$

$$\sum_{r=0}^{\mu} \frac{\Gamma(n+r+\frac{1}{2})(-\sqrt{2}x)^r\, W_{-\frac{(n+\nu+r)}{2},\, \frac{(n-\nu+r)}{2}}(\tfrac{1}{2}x^2)}{(\mu-r)!\; r!\; \Gamma(2n+r+1)},$$

then $\phi(x)$ and $\psi(x)$ are Hankel Transforms of each other provided that μ is a positive integer, $2n+1>0$, $R(2\nu+1)>0$ and $R(2n-\nu-\frac{1}{2})>0$.

G. S. Diwan, Bombay.

The "Unique Particular Integral" in the theory of ordinary linear differential equations.

In this paper are discussed the difficulties in fixing on any particular solution of $f(D)\, y = X$ as the "unique particular integral" and the advantages resulting therefrom in the treatment of specific problems. The purposes for which this conception of uniqueness are introduced are clarity in reasoning, and simplification of the proofs of certain propositions. Examples are given to show how these objects are achieved.

S. Minakshisundaram, Madras University.

On Expansion in Eigenfunctions

Let $\omega_n(x, y)$ be the normal orthogonal eigenfunctions with the corresponding eigen-values $\lambda_n > 0$, $n=1, 2, \ldots$ arranged in increasing order of magnitude, of the boundary value problem

$$\Delta\omega + \lambda^2\omega = \frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} + \lambda^2\omega = 0 \qquad \ldots \quad (1)$$

$$\omega\,(x, y) = 0 \text{ on C} \qquad \ldots \quad (2)$$

C being the boundary of a bounded regular domain D.

If $f(x, y)$ is an arbitrary function whose square is integrable L in D and which vanishes on C, and if

$$f(x, y) \sim \sum a_n\, \omega_n\, (x, y) \qquad \ldots \quad (3)$$

$$a_n = \iint\limits_{D} f\, \omega_n\, dx\, dy \qquad \ldots \quad (4)$$

the relationship between the behaviour of $f(x, y)$ in the neighbourhood of an interior point of D and summability by typical means of the series on the right of (3) is studied.

As a particular instance it might be mentioned that:

At a point of continuity of $f(x, y)$, $\sum a_n\, \omega_n$ is summable (λ, k), $k>3/2$, to the value of the function and at a point where $\sum a_n\, \omega_n$ is convergent

$$\lim_{r\to 0} f_r\,(x, y) = \lim_{r\to 0} \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} f(x+\rho\cos\theta,\, y+\rho\sin\theta)\, \rho\, d\rho\, d\theta$$

exists and is equal to the sum of the series.

Prithvi Nath Sharma, Hindu College, Delhi.

On a hystero-differential equation

The general solution of the hystero-differential equation $f^n(x) = f(1/x)$ is obtained in this paper. Its particular case of $n=1$ has been recently discussed by Silberstein (*Phil. Mag.*, 30 (1940), 185–7).

Santi Ram Mukerjee, Allahabad.

On the equation $f^r(x) = f(\pm 1/x)$

The solutions corresponding to $r=1$ to $r=6$ have been studied as also the behaviour of the solution in the general case.

A. A. Krishnaswami Ayyangar and W. F. Beard, Mysore.

A genetic problem in geometry

For some time, the authors have been tackling a genetic problem in geometry connected with the following well known result:

ABC is a triangle with O as its circumcentre. PQR is a triangle inscribed in ABC so that O is its orthocentre. Then, this triangle has one of two dissimilar properties:

(i) it is self-polar w. r. t. the circumcircle of ABC;

(ii) it is similar to ABC.

The existence of the alternatives is easily accounted for by a certain hyperbola which Mr. Beard has discovered geometrically. But why these twins should differ so radically from each other in their geometrical character, is not yet satisfactorily explained.

The following general problem is solved:

Given two points O, P and two fixed straight lines AB, AC, to find two other points Q, R on AB, AC respectively such that PQR may have O as its orthocentre.

Incidentally it is noted that if the base of a triangle is fixed and the vertex describes a curve of the n-th degree, the locus of the ortho-centre is, in general, a curve of the $2n$-th degree, of which the extremities of the base and the point at infinity in the direction perpendicular to the base are n-ple points.

S. M. Kerawala, Muslim University, Aligarh

1. *On the Frégier Forms associated with Conics*

The aim of this paper is to probe into the full implications of the well-known theorems of Frégier in the theory of conic sections. A number of interesting loci and envelopes are thus obtained, and

their intimate relationship with the theory of inversely similar figures is studied. In the light of these investigations the porism of Poncelet is finally examined for particular cases.

2. *On a Transformation for duplicating the condition of Poncelet's Porism*

The two conics of the problem are assumed to have been reduced by a homographic transformation to two circles of radii r and R with their centres at a distance d apart. Chaundy has proved that in order to convert the porism $P(2n+1)$ into $P(4n+2)$ of the separable case, it is sufficient to interchange R and d. In this paper, it has been proved that any $P(n)$ is converted into $P(2n)$ if in $P(n)$ be substituted $(R^2-d^2)^2$ instead of R, $2r^2(R^2+d^2)-(R^2-d^2)^2$ instead of r and $4r^2 Rd$ instead of d.

K. Rangaswami Aiyar, Annamalainagar.

On the space representation of Hart Tetrads.

We represent the circles of a plane by the points of a projective [3], so that the point-circles are represented by the points on a quadric Q, the same symbols denoting circles of the plane and also their representative points in [3].

Let $abcd$ be a Hart tetrad δ, $A_1B_1C_1D_1$ its complementary Hart tetrad and ABCD, the polar tetrad Δ of δ with respect to Q. Also let $\bar{Q}$ be the harmonically conjugate quadric of Q with respect to Δ H the unique quadric in the pencil determined by Q and $\bar{Q}$ circumscribed to Δ, Δ_1 the unique tetrad inscribed in H and desmic to Δ and Δ_2 the tetrad which with Δ, Δ_1 forms a desmic system. Further let us denote by Γ_1 the desmic transformation whose singular points are the vertices of Δ_1 and fixed points the vertices of Δ, Δ_2.

It is known that $\bar{Q}$ is a pair of planes intersecting in a line l and also that a, b, c, d, A_1, B_1, C_1, D_1 lie on a twisted cubic L.

We prove that: (i) A_1 B_1 C_1 D_1 are the vertices of Δ_1 and (ii) the transform of l in the cubic transformation Γ_1 is the twisted cubic L. (i) is true even when A_1, B_1, C_1, D_1 are the circles which cut a, b, c, d *all* at the same angle.

O. N. Srinivasiengar, Engineering College, Bangalore

On the Quartic Developable

It is shown that the first polar surface of any point on the twisted cubic forming the edge of regression of the developable is a ruled cubic of Cayley's type. The family of Cayley's cubic scrolls obtained as the first polars of various points on the twisted cubic possesses some interesting properties which are worked out in this paper. The nature of the first polars of other points on the developable is also investigated.

V. Rangachariar, Science College, Patna

On Systems of Quadrics in a [4]

Three quadrics in a [4] intersect in an octavic curve, through which a doubly infinite system—a net—of quadrics can be drawn. Each quadric is made to correspond to a point in a plane σ. The cones of the net correspond to a certain quintic curve without double points. The locus of the vertices of these cones is a curve. Most of the properties related to the corresponding three dimensional problem in W. L. Edge's paper (*Proc. Lond. Math. Soc.* 1931) is extended to the present case. The properties of a set of quadrics consisting of a triply infinite system through 16 associated points is also discussed.

Ram Behari, St. Stephen's College, Delhi

A Property of the Principal Ruled Surfaces through a line of a Normal Rectilinear Congruence

In a previous paper (published in the *Proc. Ind. Acad. Sc.* Vol. XII, 1940, pp. 205-207) I proved analytically the theorem that the curves on the sphere representing the distributive ruled surfaces through a line of a normal rectilinear congruence are isometric The object of this paper is to show analytically that the same result is true for the principal ruled surfaces also.

A. Narasinga Rao, Annamalai University.

On an unusual metric defined by line elements in the inversive plane.

Two line elements in the plane of inversion define an invariant angle, namely that at which the 2 circles one through each element

and the point of the other element cut each other. Some of the peculiar features of this invariant of inversive geometry are studied here. Essentially it is the study of an invariant defined by 2 tangent lines to a quadric in a projective S_3 under the group of projective transformations carrying the quadric into itself.

Mohammad Shabbar, Aligarh

Continuous groups of affine collineations and local applicabilities and their applications to Lorentz-invariant Riemann Spaces.

I study here the groups of affine collineations and local applicabilities in Riemannian spaces admitting the Lorentz-group L_6 whose generators are.

$$x^2\frac{\partial}{\partial x^1}+x^1\frac{\partial}{\partial x^2};\quad x^3\frac{\partial}{\partial x^1}+x^1\frac{\partial}{\partial x^3};\quad x^4\frac{\partial}{\partial x^1}+x^1\frac{\partial}{\partial x^4};$$

$$x^3\frac{\partial}{\partial x^2}-x^2\frac{\partial}{\partial x^3};\quad x^4\frac{\partial}{\partial x^3}-x^3\frac{\partial}{\partial x^4};\quad x^4\frac{\partial}{\partial x^2}-x^2\frac{\partial}{\partial x^4}.$$

The spaces have been classified according to the groups they admit and the properties of these groups have been studied.

R. Vaidyanathaswamy, Madras University.

Related topologies.

K. Suryanarayanamurthi, Madras University

On the problem of Metrisation in neighbourhood spaces

The object of this paper is to examine the equivalence of the two sets of axioms—Hausdorff axioms and Kurutowski axioms, for a Topological space and analyse in particular the conditions for metrisations of Neighbourhood spaces. The following results are proved :—

I. A T_1—space satisfying the first countability axiom is metrisable if and only if it is completely regular.*

II. A T_1—space with equi-continuous functions satisfies the first countability axiom.†

* A Space is completely regular if, when P and Q are any two closed sets in it of which P is a single point set (p), there exists a continuous function $f(p)$ such that $f(p)$ is equal to 0 for $P \in (p)$, $f(p)=1$ for $p \in Q$ and $0 \leqslant f(p) \leqslant 1$ throughout.

† If given an ε there is a neighbourhood U such that the oscillation of each function f of the family F in U is less than ε, then all those functions of F are said to be equi-continuous.

By the second result it is shown that the theorem of metrisation of Neighbourhood Spaces, due to Chittenden ‡ in terms of equi-continuous functions is the same as the result I.

C. N. Srinivasiengar, Engineering College, Bangalore.

The Resultant of wrenches on two given screws

The cylindroid associated with a pair of given screws of arbitrary position is worked by considerations which are more direct than those in the familiar method of Ball's *Theory of Screws.* A number of new results regarding the properties of the cylindroid are also worked out.

B. R. Seth, Hindu College, Delhi.

Transverse vibrations of triangular membranes.

Besides the known solutions of a right-angled triangle and that of an equilateral triangle new results have been obtained for

(i) an isosceles triangle containing an angle of 120°;

(ii) a right angled triangle containing an angle of 60°.

The frequencies in the known cases are given by

$$\frac{p}{2\pi} = \frac{c}{4a}\,[(2m+1)^2+(2n+1)^2]^{\frac{1}{2}}, \qquad \frac{p}{2\pi} = \frac{c}{a}\,[m^2+n^2+mn]^{\frac{1}{2}}.$$

In the case of (i) and (ii) it is found that

$$\frac{p}{2\pi} = \frac{c}{2a}\,[(2m+1)^2+(2n+1)^2+(2m+1)(2n+1)]^{\frac{1}{2}},$$

where c and a are known constants, and m and n are integers.

K. Nagabhushanam, Andhra University, Waltair.

Subgroups of the infinitesimal Contact Transformations

The following subgroups of the general infinitesimal contact transformations of $p_r\, dq^r - Hdt$, are considered:

(i) those confined to a surface which is an integral of Hamilton's equations, such as surfaces of constant energy;

‡ *Bulletin of the American Mathematical Society* Vol. 33; 1927.

(ii) those displacing points on a trajectory into a neighbouring one on the same trajectory;

(iii) those for which $dW=0$, where

$$p_r\, dq^r - \mathrm{H}dt = (p')_r\, d(q')^r - \mathrm{H}'\, dt' + d\mathrm{W}.$$

From (i) and (iii), further subgroups can be derived; but (ii) and (iii) mutually exclusive, and no further subgroups are obtainable.

Ram Ballabh, Lucknow University.

Superposable Fluid Motions

The paper deals with superposable fluid motions of which the vortex lines coincide with stream lines. It has been established that if λ, the proportionality factor between the vorticity and velocity components, be a constant, the stream lines of the motions can lie on concentric spheres, coaxial cylinders and parallel planes but not on coaxial planes or coaxial cones having a common vertex. Other results have been obtained regarding λ being a function of space variables and t.

G. R. Seth, Hindu College, Delhi.

Motion of a parabolic cylinder through a viscous liquid.

The resistance suffered by a parabolic cylinder in a non-viscous liquid of density ρ is $\pi\rho a\mathrm{U}^2$ where U is the uniform velocity and $4a$ is its latus-rectum, provided the pressure at infinity is taken to be zero.

When the liquid is viscous and we neglect the inertia terms, we get a solution of the problem in which the velocity at infinity becomes as great as $\sqrt{r}$. In the corresponding solution of a circular cylinder the velocity at infinity varies as log r. But we do not get a finite value for the drag.

On applying Oseen's approximation to the problem all the boundary conditions, including the one at infinity, are satisfied, but again no greater value of the drag is obtained.

S. R. Das, Delhi.

1. *Effect of shifting the Centre of Pressure in a Double-lifting Aeroplane.*

In this paper it has been shown by considering the equations of motion that in the general case of a double-lifting aeroplane where

neither the front nor the rear planes are neutral and when the machine is moving uniformly the shifting of the centre of pressure will have the effect of either depressing or elevating the flight-path.

2. *The Steering of an Aeroplane in a horizontal circle.*

Dr. H. Reissner tackled the problem of lateral steering in 1910–11 and it was then tackled by Dr. G. H. Bryan. The present paper is a further investigation on the subject. By considering the equations of motion in the case of a symmetrical aeroplane such as commonly exists in practice, relations between the propellar thrust and the components of the air resistances have been established.

H. K. Sen, Allahabad.

Polytropic Gas Spheres with Variable Index

A sphere of perfect gas is considered, whose polytropic index n, defined by $\frac{d}{dr}(\log P) = \left(1 + \frac{1}{n}\right)\frac{d}{dr}(\log \rho)$, P and ρ being respectively the pressure and density at a distance r from the centre, varies in any manner from shell to shell between the limits n_1 and n_2. It is shown from quite general considerations that quite a number of properties of the variable polytrope, besides the gravitational potential energy, attain their extremal values in the limiting uniform polytrope of index n_1 or n_2. Using Candler's results in M. N. **100**, 14, 1939, fresh intermediate properties have been deduced for the variable polytrope, and alternative proofs have been given of Chandrasekhar's theorems on the central value of the ratio of the radiation to gas pressure. The polytropic index has been shown to increase monotonically from the centre of a variable polytrope to its surface and the signs of the variation of the other important physical variables have been found. Two new integral theorems have been proved on the central potential and temperature of a perfect gas sphere in gravitational equilibrium.

M. Zakiuddin, Muslim University, Aligarh.

A new method for evaluating the Vibration Constants of a Diatomic Molecule

The constants associated with a diatomic molecule can be calculated after the wave numbers of band heads, emitted by these molecules, have been fitted in a quantum vibrational scheme.

The equation is

$$\nu = \nu_e + [w_e'(v'+\tfrac{1}{2}) - x_e w_e'(v'+\tfrac{1}{2})^2] - [w_e''(v''+\tfrac{1}{2}) - x_e w_e''(v''+\tfrac{1}{2})^2]$$

and according to the old quantum theory we get for 0,0 the sequence of band heads,

$$\nu_{\text{head}} = \nu_0 + (a'-a'')\, v'' - (b'-b'')\, v''^2 + (c'-c'')v''^3$$

For 0, 1 sequence,

$$\nu_{\text{head}} = (\nu_0 - a' - b' - c') + (a' - a'' + 2b' + 3c')v'' - (b' - b'' + 3c')v''^2 + (c' - c'')v''^3.$$

where,

ν_{head} is wave number for 0,0 band head,

v'' is the final vibration quantum number,

a' and a'' are vibrational constants,

b' and b'' are anharmonic constants,

c' and c'' are correction terms for anharmonicity in the initial and final states.

The method of Birge and Shea is generally employed to get quadratic and cubic equations to represent the band heads mathematically. Recently Mr. Sulaiman Kerawala of the Aligarh Muslim University has devised a new method for fitting polynomials. This method can be employed to fit the wave numbers instead of the old method.

Bands of the calcium fluoride molecule have been fitted by me in mathematical scheme using both methods and old quantum mechanics and the superiority of the second method over the first is shown. New Quantum mechanical calculations can also be made.

M. V. Jambunathan, Maharaja's College, Mysore.

1. *Probability of the* " *Obtuse* "

In an article contributed to *The Mathematical Gazette*, (October 1939) Mr. C. O. Tuckey mentions, without proof, that the probability of drawing an obtuse-angled triangle at random varies from ·57 to ·75. In the present paper various values of the probability are deduced based on different hypotheses about "equally probable" cases.

2. *Mathematical Measure of a Favourable Set of Group Minima.*

In this paper an attempt is made to indicate mathematically in what manner the chances of securing a pass in an examination by a candidate who can score any percentage between 30 and 45 are enhanced by the introduction of a particular set of group minima.

A. A. Krishnaswami Ayyangar, Maharaja's College, Mysore.

1. *An improved formula for the Coefficient of Contingency.*

In statistics, the coefficient which summarises the mutual dependence of two attributes of the elements of a universe is called a coefficient of contingency. Two such coefficients C and T due respectively to Pearson and Tschuprow are well known. T is usually considered as an improvement on C. But we remark that T does not remedy the defect in C in a $s \times t$—fold classification when $s \neq t$. We propose a better coefficient K, which is related to C and T by the equations:

$$K^2 = \frac{c^2}{(1-c^2)(t-1)} = \frac{T^2\sqrt{(s-1)}}{\sqrt{(t-1)}},\ (s > t)$$

K has the advantage of a unique upper limit, unity, which can be actually reached without making the classification too fine and thus introducing errors of sampling.

It is shown that the maximum value of T is $\sqrt[4]{\left(\frac{t-1}{s-1}\right)}$ and so, by a proper adjustment of t with respect to s, the value of the coefficient can be made less than any assigned small value. T will thus give spurious results.

2. *Fiducial Probability.*

There are two theories in Statistical Estimation, one named Fiducial Probability by R. A. Fisher, who is peculiarly fond of using ordinary words in extraordinary senses, and the other started by E. B. Wilson under the notion of Confidence Intervals. This paper attempts at a clarification of the relative merits of the ideas underlying these two almost synonymous names.

3. *Incompatibility of k-statistics.*

Tschuprow called G the presumptive value of a statistic F when the mathematical expectation of G is equal to F. In 1927, Mr. Bertelson worked out, according to Tschuprow's definition, the presumptive values of the semi-invariants, and these were introduced

later by Fisher as his k-statistics. By a very complicated method, Mr. Bertelson has proved the incompatibility of k_4, i.e., it does not correspond to any real set of observations. We now give a simple proof of this incompatibility, when for a sample of size n, β_2 is less than $(n^2+4n-9)(n^2-1)$.

4. *Conventions in Mathematics.*

There are two types of conventions, (1) linguistic, and (2) logical. Assigning numbers to persons and things belongs to the former type, while the latter is implied in 'plus' and 'minus'. A convention does not possess the 'naivete' of an axiom that disarms suspicion. Custom is a blind unthinking adherence to something, which may be the result of a convention, whereas a convention is the result of a conscious and deliberate preference. The conventions 'plus' and 'minus' reflect most appropriately an essential dualism in nature. In mathematics, they represent dimensional directions.

Conventions are not mere peace-makers. They are fruitful stimuli for the advancement of thought. They led Mobius to the discovery of solids without volumes.

Very few elementary mathematical text-books formulate clearly and explicitly the most useful conventions, nor are they consistent in the adoption of conventions. At the threshhold of analytical geometry, we find the positive side of a line determined in at least three different ways. As regards the standard form of the equation of a line, Salmon and Askwith make the co-efficient of y positive, and Casey makes the absolute term positive, while recently Mr. J. M. Child makes the coefficient of x positive. It is desirable that we standardise one of these conventions, at least in the interests of students taking public examinations.

Another useful convention which is worth universal adoption is to regard a triangle not in the finite and restricted manner of old Euclid but in the more comprehensive way of trigonometry, in which the sides and angles are determined only by the well known sine-rule and angle-sum properties. The advantages of this neglected principle were pointed out by H. W. Richmond and H. Lob a dozen years ago but nobody seems to have taken the matter seriously.

5. *Inequalities and Unemployment.*

The paper starts with the paradoxical notions of inequality which are prevalent among economists. It introduces the concept of

'symmetric groups' of income-receivers and proposes a mathematical criterion to distinguish between over-employment and under-employment based on an empirical quadratic law, $k(x-a)(x-b)$, which seems to be eminently fitted for the consideration of the inequalities in question. Mathematics apart, it is suggested that a reform based on psychological and ethical lines will sooner solve the problem of inequalities than any purely economic measures.

M. R. Doresamiengar, Maharaja's College, Mysore

Parameters of 'Personal Distribution'

The study of parameters is important in any branch of Economics and particularly in 'personal distribution.' Thus levels of incomes are determined not only by the number of able-bodied men but also by the degree of industrialisation, volume of trade. facilities for migration from employment to employment and other factors, assessable and non-assessable. The analytical laws of incomes *in vogue* may be taken as the spade work in the field and it is proposed to work out the feasible parametric forms on the lines of Moore's 'Synthetic Economics.' Dr. H. Hotelling's work on parameters in Statistics is the chief source of inspiration of this paper.

Hansraj Gupta, Government College, Hoshiarpur

The Calendar

In this paper the principles on which the Christian, Vikrama and Hijri calendars are based are explained and the nature of the corrections required in each case discussed.

R. Krishnamurti, Hyderabad

Some facts about the Saros and the Metonic cycle.

In this paper it is discussed whether the Saros and the metonic cycle could have been the results of calculation instead of a purely observational result as it is commonly supposed.

K. R. Gunjikar, Bombay.

Some observations on the February comet.

SYMPOSIUM ON GROUP THEORY

Report on the Combinatorial Theory of Groups

BY

Prof. F. W. LEVI, (*Calcutta*)

Opening the general discussion on theory of groups, Prof. Levi pointed out that the automorphisms of a system of any kind form a group, and that every group can be considered as a group of automorphisms. This inter-connection seems to be the reason for the central position of the theory of groups. Starting from the idea of automorphism,—*i. e.* of generalised permutations—a review was given of the development of the theory in recent times.

Besides this main-line of progress, a certain by-way has got enhanced importance, where the idea of *combination* prevails. This theory is the subject of the report. The elements of a group are constructed by means of *generators* $a, b, c, \ldots$ which may be finite or infinite in number. With the generators and their *inverse* symbols a^{-1}, b^{-1}, c^{-1}, ... one forms *words* by arbitrary combination, e.g. ab, bb, $a\ b\ a^{-1}\ b^{-1}$, ...; these words represent group-elements, and their multiplication is effected by putting together the corresponding words. Such words like aa^{-1}, $a^{-1}a\ \ bb^{-1}$, ... are called *trivial relations*; they stand for the unit element 1. By striking out and inserting trivial relations, the group-element is not altered; using these operations, one can transform every word either into 1, or into a word which is free from trivial relations. These words and the symbol 1 are called reduced words. One can prove that they form a group which is called the *free group* F generated by a, b, c, ... Let N be a normal subgroup of F and $G = F/N$; consider the class a of elements of F which by the homomorphism $F \rightarrow G$ are mapped on the same element of G. Starting from any element of a, one gets all the other ones by repeated inserting and striking out of words which represent elements of N (in particular trivial relations; since they represent 1); these words are called the *relations* in G. One can obtain all of them from a certain subset, the *generating relations*, by successive composition and transformation. Every group is homomorphic to a free group; thus it can be constructed by generators and generating relations. It is therefore uniquely determined by a free group F and a normal subgroup in it. This representation is

not unique, since e. g. F generated by a, b, c, ... is also generated by a, ba, c, ... but the number (finite or transfinite) of the free generators of F is an invariant. Let F, generated by n free generators, admit a representation by m generators with q generating relations, and let q be the minimal number for those generators, then $m=n+q$. If F′ is a subgroup of F, then F′ is a free group. The numbers n and N of the free generators of F and F′ and the index j of F over F′ are interconnected by $j+N=1+nj$. This formula determines N when j is finite but not when j is infinite. If j is infinite, and F′ a normal subgroup, $N=nj$ holds. From this formula is easily seen that free groups with 2 (or more) generators have subgroups with an enumerable (and therefore also with any finite) number of generators. The words where every generator occurs as often as its inverse symbol, represent the elements of the commutatorgroup $C_1(F)=(F, F)$ of F. There exists an infinite sequence $C_2(F)=(C_1, F)$; $C_3(F)=(C_2, F)$ ... of different commutatorgroups (unless F has a single generator) which have 1 as the only common element. The homomorphism $F \to G$, maps $C_m(F) \to C_m(G)$, but the commutatorgroups of G may not all be different. In this way the theory of free groups is applied to other groups. The methods used in the combinatorial theory of groups can be classified into (i) combinatorial, (ii) group theoretical, (iii) topological methods.

1. Operating with combinatorial methods, one considers what occurs when a number of words are combined and relations are inserted and struck out; the simplifications which can be effected by a change of the system of generators are important. These investigations are often tricky, but it seems that they cannot be completely avoided. The most important auxiliary notion is the "length" (i. e. the total number of symbols) of a word, the length is mostly used as a perameter for mathematical induction.

2. Group theoretical methods can be used as the theory advances and group theoretical results are obtained. The most fundamental operation is the homomorphism in particular the homomorphism "making G abelian", i.e. mapping G on the factorgroup of the commutatorgroup.

3. The topological methods are combinatorial methods where the combinations on a linearly ordered set are replaced by combinations on a manifold of more dimensions. This way of procedure tallies better with the nature of the theory since the "relations" are invariant for cyclic permutations. The groups represented by

generators and generating relations admit a topological representation by "group-images" of various kinds, and these can be treated by methods of the combinatorial topology. The report dealt chiefly with the group-images and with the theory of the free products in which all the different methods of combinatorial theory of groups are used. As an important application, the theory of knots and *chains was mentioned.*

Theory of p-groups

BY

M. A. JABBAR, M. Sc., (*Calcutta*).

Every Abelian group of finite order is the direct product of characteristic subgroups which are p-groups (i. e, groups of prime power orders). Thus the study of Abelian groups reduces to that of Abelian p-groups. These have a basis representation which is unique (in the sense of isomorphism). The structure of non-Abelian groups is more complicated, but there exists some important similarities with the Abelian case, which offer an opportunity for starting a separate investigation for them. Burnside has shown[1] that every p-group has a minimal basis, which is not unique. In his paper, Hall has shown[2] that there is a class of p-groups (termed regular), which admit unique basis representation similar to Abelian p-groups. For every exponent n there exists only a finite number of p-groups of order p^n (p running over all the prime numbers) which are not regular, whereas the number of regular p-groups is infinite.

If H and K are any two groups, then (H, K) is the group generated by elements of the form $(h, k) = h^{-1} k^{-1} hk$, where h is an element of H and k is an element of K. The series $G = H_1 > H_2 > \cdots\cdots > H_c > H_{c-1} = 1$, where $H_i = (H_i - 1, G)$, exists for any p-group G (more generally for any nilpotent group)[3] and is known as the lower central series. The length c of the series is called the class of the group.

An element P of a group G is said to be a commutator of weight 1 and $(P, Q) = P^{-1} Q^{-1} PQ$ is a commutator of weight 2; generally if S is a commutator of weight w_1 and T is another commutator of weight w_2, then the complex commutator (S, T) is

[1] *Proc. Lond. Math. Soc.* 1913.

[2] *Proc. Lond. Math. Soc.* 1934.

[3] *H. Zassenhaus*: *Lehrbuch der Gruppentheorie,* 1. Bd. IV, §5.

said to be a commutator of weight w_1+w_2. The generators of H_i in the series given above are of weight i and the subgroup H_i is a complex commutator group of weight i. If $f(G)$ is any complex commutator subgroup of a p-group G and if it is of weight w, then $f(G) \leqslant H_w$. In particular if w is $>c$, the class of G, then $f(G)=1$.

If P and Q are any two elements of a group G, then for any exponent α and any pre-assigned integer m

$$(PQ)^{p^\alpha} = P^{p^\alpha} Q^{p^\alpha} R_3^{n_3} R_4^{n_4} \ldots\ldots\ldots R_r^{n_r} T_m.$$

where R_i's: are various complex commutators in P and Q arranged in order of non-decreasing weights, otherwise arbitrary;

T_m belongs to H_m;

and n_i's are positive integers depending on p^α and the weight w_i of R_i.

In the case of a p-group (more generally a nilpotent group) if the weight of R_{r+1} is greater than the class of the group generated by P and Q, then R_{r+1} and $R_{s>r+1}$ are all equal to 1. Hence in these cases

$$(PQ)^{p^\alpha} = P^{p^\alpha} Q^{p^\alpha} R_3^{n_3} \ldots\ldots\ldots R_r^{n_r}.$$

The groups for which $n_i=p^\alpha$, for $i=3, 4, \ldots\ldots r$, are said to be "regular" p-groups of this kind have the characteristic property that for every exponent α, $P^{p^\alpha} Q^{p^\alpha} = R^{p^\alpha}$, for a suitable element R of the group and that $P^{p^\alpha} = Q^{p^\alpha} = 1$ implies $(PQ)^{p^\alpha} = 1$. So the p_α th powers of the elements of regular group G form a characteristic subgroup $\mho_\alpha$ and that the elements which satisfy the relation $P^{p^\alpha}=1$, form another characteristic subgroup Ω_α, for $\alpha=0, 1, 2, \ldots$ $\ldots\ldots \mu$, where p^μ is the highest order of the elements of G.

If p^{ω^α} is the order of the factor group $\Omega_\alpha/\Omega_{\alpha-1}$, then $\omega_1 \geqslant \omega_2 \geqslant \ldots \geqslant \omega^\mu$ holds; and if $\mu_\beta =$ the number of $\omega_\alpha \geqslant \beta$, then μ_i's are invariants for G and $p^{\mu_1}, p^{\mu_2}, \ldots\ldots\ldots p^{\mu_{\omega_1}}$ are the orders of the basis elements $P_1, P_2, \ldots\ldots\ldots P_r$ respectively. So the group may be represented by the invariants $(\mu_1, \mu_2, \ldots\ldots \mu_{\omega_1})$ which is known as the type of the group as in the case of Abelian groups of prime power orders. But though to every type $(\mu_1, \mu_2 \ldots\ldots \mu_r)$ there exists only one Abelian group (one in the sense of isomorphism), it is not generally so in the case of regular p-groups.

Characters of the Symmetric Group

BY

DR. M. ZIAUD DIN, *Lahore*

The characters of the symmetric group, play an important part in higher algebra, statistics, quantum mechanics and nuclear physics. Investigations on this subject have been carried out chiefly by Frobenius, Littlewood and Richardson, Murnaghan and M. Ziaud Din.

The author has published the tables of group characters of the symmetric group of degrees 11, 12 and 13 in the *Proceedings London Math. Soc.* (2) 39 (1935) and 42 (1937), and the tables of degrees 1—10 have been published by other authors quoted above.

Frobenius was the first who took up the construction of tables of group characters using the following method. Let S be an operation of the symmetric group on n symbols, containing α_1 cycles on one symbol, α_2 cycles on two symbols and so on, such that $n = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots$; the numbers $\alpha_1, \alpha_2, \alpha_3, \cdots$ define the class ρ to which S belongs and the coefficient of $\pm\ x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$, will give the character of S in an irreducible representation defined by the sequence $\lambda_1, \lambda_2, \lambda_3, \cdots \lambda_n$ in the product

$$(x_1 + x_2 + \cdots x_n)^{\alpha_1}(x_1^2 + x_2^2 + \cdots\ )^{\alpha_2} \cdots (x_1^n + x_2^n + \cdots)^{\lambda_n} \cdot A(x_1\, x_2 \cdots x_n).$$

where $A(x_1\ x_2 \cdots)$ denotes the simple alternant.

Frobenius method is very laborious. Simpler and quicker methods have been developed by Littlewood and Richardson, and Murnaghan, a detailed account of which will be found in D. E. Littlewood's Book 'The theory of group characters (1940)' and in Murnaghan's "The theory of group representation" (1938). The methods of Littlewood and Richardson, are based on Immanants and S-functions, which may be briefly described as follows.

Immanants of a Matrix

Let $[a_{st}]$ be a matrix of order n^2, S being any permutation $e_1, e_2, e_3 \cdots e_n$ of the numbers 1, 2, 3, $\cdots n$ and

$$P_S = a_{1e_1} \cdot a_{2e_2} \cdot a_{3e_3} \cdots a_{ne_n}.$$

There are $n!$ products P_S corresponding to the $n!$ permutations of the symmetric group.

If $\chi^{(\lambda)}$ denote the character of the symmetric of order $n!$, corresponding to the partition $(\lambda)\equiv(\lambda_1, \lambda_2, \cdots \lambda_p)$ of n, the immanant of the matrix $[a_{st}]$ corresponding to the partition (λ) is defined by

$$|a_{st}|^{(\lambda)} = \sum \chi^{(\lambda)}(S) \cdot P_S.$$

the summation being with respect to the $n!$ permutations of the symmetric group.

Since the character $\chi^{(n)}$ is unity for every operation of the group, and the character $\chi^{(1^n)}$ is $+1$ for a positive permutation and -1, for a negative permutation, the permanent and determinant of a matrix are special cases of immanants. Thus

$$\overset{+}{|}\, a_{st} \,\overset{+}{|} = |a_{st}|^{(n)}$$

$$|a_{st}| = |a_{st}|^{(1^n)}.$$

S-Functions

These are called S-functions after Schur, who first dealt with symmetric functions in relation to groups. S-functions may be defined from the immanant of a matrix as follows:

If (λ) is a partition of r, with the parts in descending order, the S-function denoted by $\{\lambda\}$ is defined by the equation

$$r!\,\{\lambda\} = |z_r|^{(\lambda)}$$

where the matrix $[z_r]$ is

$$[z_r] = \begin{vmatrix} S_1, & 1, & 0 & 0 & \cdot & 0 \\ S_2, & S_1, & 2 & 0 & \cdot & 0 \\ S_3, & S_2, & S_1 & 3 & \cdot & 0 \\ S_{r-1} & \cdot & & \cdot & \cdot\ S_1, & r-1 \\ S_r, & \cdot & & \cdot & \cdot\ \cdot & S_1 \end{vmatrix}; \quad S_r = \sum x_1^r. \quad \text{(power-sum, symmetric function).}$$

For a symmetric group of order $r!$, if ρ denotes the class of order h_ρ, and $\chi_\rho^{(\lambda)}$ the character for the class ρ, then S-functions satisfy the relations.

$$(1) \quad r!\,\{\lambda\} = \sum h_\rho\, \chi_\rho^{(\lambda)}\, S_\rho$$

$$(2) \quad S_\rho = \sum \chi_\rho^{(\lambda)} \{\lambda\}.$$

where $S_\rho \equiv S_1^{\alpha_1} S_2^{\alpha_2} S_3^{\alpha_3} \ldots$

S-functions can also be expressed as quotients of two alternants, *i.e.* general alternant ÷ simple alternant. In this form these functions have been already dealt with by several writers, as has been described by the author in the paper on 'Determinantal Symmetric Functions' *Proc. Edinburgh Math. Soc.* (2) 4 Part I 47.

Tables of the characters of the symmetric group can thus be constructed using the properties of S-functions given in Littlewood's book. The characters can be checked by using the orthogonal properties

$$\sum \chi_\rho^{(\lambda)^2} = h/h_\rho$$

$$\sum \chi_\rho^{(\lambda)} \chi_{\rho'}^{(\lambda)} - 0$$

where h is the order of the group.

S-functions and group characters are also applicable to Invariant matrices as I have shown in my paper 'Invariant matrices and S-functions' *Proc. Edin. Math. Soc.* (2) 5 Part I 43.

Symposium On Fourier Integrals and Transforms

BY

DR. R. S. VARMA, *Lucknow University.*

1. *Introduction.* The theory of Fourier Integrals is closely allied to the theory of Fourier series. A function $f(x)$, of period $2\pi\lambda$, is represented by the Fourier series

$$f(x)=\tfrac{1}{2}\,a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{nx}{\lambda}+b_n\sin\frac{nx}{\lambda}\right), \qquad \ldots \quad (1\cdot1)$$

the coefficients a_n and b_n being given by

$$a_n=\frac{1}{\pi\lambda}\int_{-\pi\lambda}^{\pi\lambda}f(t)\cos\frac{nt}{\lambda}\,dt, \qquad \ldots \quad (1\cdot2)$$

$$b_n=\frac{1}{\pi\lambda}\int_{-\pi\lambda}^{\pi\lambda}f(t)\sin\frac{nt}{\lambda}\,dt. \qquad \ldots \quad (1\cdot3)$$

With the aid of (1·2) and (1·3), (1·1) may be written as

$$f(x)=\frac{1}{2\pi\lambda}\int_{-\pi\lambda}^{\pi\lambda}f(t)\,dt+\sum_{n=1}^{\infty}\frac{1}{\pi\lambda}\int_{-\pi\lambda}^{\pi\lambda}f(t)\cos\frac{n(x-t)}{\lambda}\,dt \ \ldots \quad (1\cdot4)$$

If we put $\dfrac{n}{\lambda}=u$, $\dfrac{1}{\lambda}=\delta u$, and make λ tend to ∞, the sum on the right of (1·4) passes formally into the form

$$f(x)=\frac{1}{\pi}\int_0^{\infty}du\int_{-\infty}^{\infty}f(t)\cos u(x-t)\,dt \qquad \ldots \quad (1\cdot5)$$

This is *Fourier's Integral Formula.*

If $f(x)$ is an even function, Fourier's integral becomes

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\cos xu\,du\int_0^{\infty}\cos ut\,f(t)\,dt, \qquad \ldots \quad (1\cdot6)$$

the term involving $\sin ut$ vanishing identically. This is *Fourier's cosine formula.* If now we write

$$F_c(u)=\sqrt{\left(\frac{2}{\pi}\right)}\int_0^{\infty}\cos ut\,f(t)\,dt, \qquad \ldots \quad (1\cdot7)$$

then (1·6) gives

$$f(x)=\sqrt{\left(\frac{2}{\pi}\right)}\int_0^{\infty}\cos xu\,F_c(u)\,du, \qquad \ldots \quad (1\cdot8)$$

There is therefore a reciprocal relation between the functions $f(x)$ and $F_c(x)$. A pair of functions connected by formulae (1·7) and (1·8) are known as *Fourier cosine transforms* of each other.

Similarly from (1·5), if $f(x)$ be odd, we obtain

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin xu \, du \int_0^\infty \sin ut \, f(t) \, dt, \quad \ldots \quad (1\cdot9)$$

and from this the reciprocal formulae

$$F_s(u) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \sin ut \, f(t) \, dt \quad \ldots \quad (1\cdot10)$$

and

$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \sin xu \, F_s(u) \, du \quad \ldots \quad (1\cdot11)$$

The relation (1·9) is known as the *Fourier's sine formula* and a pair of functions satisfying (1·10) and (1·11) are called *Fourier sine transforms* of each other.

The *exponential* form of Fourier's formula, given by Cauchy, is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-ixu} \, du \int_{-\infty}^\infty f(t) \, e^{iut} \, dt, \quad \ldots \quad (1\cdot12)$$

which leads us to the unsymmetrical formulae

$$F(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(t) \, e^{iut} \, dt, \quad \ldots \quad (1\cdot13)$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(u) \, e^{-ixu} \, du \quad \ldots \quad (1\cdot14)$$

If $f(x)$ is even, $F(x) \equiv F_c(x)$ and if $f(x)$ is odd, $F(x) = i\, F_s(x)$.

It is interesting to see that the pair of formulae, known as *Mellin's inversion formulae*,

$$F(s) = \int_0^\infty x^{s-1} f(x) \, dx,$$

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s) \, x^{-s} \, ds$$

leads us to the fact that $\sqrt{(2\pi)} \, e^{cx} f(e^x)$ and $F(c+ix)$ are Fourier transforms of each other in the sense of (1·13) and (1·14).

2. The subject of Fourier Integral is important both from the point of view of pure and applied mathematics. The so-called Laplace transform and its complex analogue now called the Fourier

transform were both known to Laplace as well as to Fourier, who used them in connection with the problem of the conduction of heat in the half plane.

Fourier series may be used in the analysis of a complex wave form such as would occur on a gramophone record. The coefficients a_n and b_n would then be proportional to the amplitudes of the various harmonics occurring as over tones in the original sound. The Fourier integral would then be the sum total of the contributions due to an irregular wave - form of all adjacent frequencies over the entire frequency spectrum. It is implied that the component frequencies exist throughout time, past and present, and, at a particular epoch, their phases are such that they describe the required wave - form.

3. In the theory of Fourier series Riemann Integration is sufficient to give all the theorems when we proceed from a given function to its Fourier coefficients, but when we go to the function from a set of coefficients, Riemann integration does not help us and we have to take into consideration integration in a wider sense, viz. that of Lebesgue. Further the Fourier coefficients are a discrete non-periodic set of numbers while the function is defined for a continuous infinity of values but is periodic and hence need only be given within a single period. In the theory of the Fourier Integral, it is not possible to segregate the difficulties into two such categories. If the function to be expanded is $F(u)$ of (1·13), then $f(u)$ plays the part of the set of Fourier coefficients. The formal similarity between (1·13) and (1·14) shows that it is not possible to say which of f and F is the function to be expanded and which is the set of coefficients. Both are non-periodic coefficients defined over continuous infinite ranges. In spite of this, the difficulties are there in the theory of Fourier Integrals and we have to construct its theory on the basis of Lebesgue integration.

4. *Functions of class* L. A direct consideration gives that if $f(t)$ is integrable in the Lebesgue sense over $(-\infty, \infty)$ and is of bounded variation in an interval including the point x, then

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\} = \lim_{u\to\infty}\frac{1}{\pi}\int_0^u du\int_{-\infty}^{\infty} f(t)\cos u(x-t)\,dt; \quad \ldots(4{\cdot}1)$$

as a particular case, if $f(t)$ is continuous,

$$f(x)=\frac{1}{\pi}\int_0^{\infty} du\int_{-\infty}^{\infty} f(t)\cos u(x-t)\,dt. \quad \ldots (4{\cdot}2)$$

An interesting result for the class of functions under considera-tion is that, if $f(t)$ is $L(-\infty, \infty)$, then the integral on the right of

(4·2) is summable (C, 1) to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ wherever this expression has a meaning.

5. *Functions of class* L^2. We have seen that though the reciprocal formulae giving $f(x)$ and $F_c(x)$ or $F_s(x)$ (the cosine or sine transforms) are symmetrical, yet the conditions which these functions satisfy are quite different. Plancherel, using the theory of mean convergence, has developed the transform theory in which he uses functions of the class L^2. Thus, if $f(x)$ belongs to $L^2(0, \infty)$, then the formulae for the cosine transform hold in the sense that, as $a \to \infty$, the integral

$$g_c(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^a \cos xt \;\; f(t)dt.$$

converges in the mean over $(0, \infty)$ to a function $g(x)$ of the class

$L^2 (0, \infty)$ and
$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^a \cos xt \;\; g(t)\, dt$$

converges in mean to $f(x)$.

We have almost everywhere

$$g(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{d}{dx} \int_0^\infty f(t) \frac{\sin xt}{t} dt,$$

and,
$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{d}{dx} \int_0^\infty g(t) \frac{\sin xt}{t} dt.$$

The corresponding theorem for sine transforms holds with $\cos xt$ replaced by $\sin xt$ and $\sin xt$ by $1-\cos xt$.

6. *Parseval formulae.* Suppose now that $g_1(x)$ and $g_2(x)$ are Fourier cosine transforms of $f_1(x)$ and $f_2(x)$, we have, both for transforms of L and of L^2,

$$\int_0^\infty g_1(x)\, g_2(x)\; dx = \int_0^\infty f_1(x)\; f_2(x)\; dx$$

and

$$\int_0^\infty \{g_r(x)\}^2 dx = \int_0^\infty \{f_r(x)\}^2\, dx \qquad r=1, 2.$$

The formulae remain valid even when $g_1(x)$ and $g_2(x)$ are Fourier sine transforms of $f_1(x)$ and $f_2(x)$ respectively.

These are analogous to Parseval's formulae in the theory of Fourier series

$$\frac{1}{\pi} \int_0^{2\pi} \{f(x)\}^2 dx = \tfrac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

7. *Self-reciprocal Functions.* The time at my disposal does not allow me to deal with the various forms of general Tauberian theorems, or the Lambert-Tauber theorem and the de la Valee Poussin-Hadamard theorem concerning the distribution of primes, or the theorem that a function which has a spectrum has a positive spectrum, or with the recent and very interesting and fruitful investigation of Titchmarsh on expansions in Eigenfunctions (*Q. J. Math.* vols. 11, 12 and *J. L. M. S.* vol. 14). But I would like to take some time more in giving a passing reference to the theory of self-reciprocal functions as this subject has been engaging my attention for the last few years. In the general sense, a function $f(x)$ is said to be *self-reciprocal in the Hankel Transform* or is said to be R_ν (using an abbreviation given by Hardy and Titchmarsh) when

$$f(x) = \int_0^\infty f(y) \sqrt{(xy)}\, J_\nu(xy)\, dy. \qquad \ldots \quad (7\cdot1)$$

For $\nu = -\frac{1}{2}$, this reduces to

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \cos xy \, dy.$$

and for $\nu = \frac{1}{2}$,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \sin xy \, dy.$$

In the last two cases, $f(x)$ is said to be *its own cosine* or *sine Transform.*

A formal solution of (7·1) is given by

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\, \Gamma\left(\tfrac{1}{2}\nu + \tfrac{1}{2}s + \tfrac{1}{2}\right) \psi(s)\, 2^{-s}\, ds,$$

where $\psi(s) = \psi(1-s)$.

The remarkable memoir of Hardy and Titchmarsh (*Quarterly Journal of Mathematics* (Oxford series), Vol. 1 (1930), pp. 196-231) is very useful in this direction.

During the last ten years, a number of formulae connecting different classes of self-reciprocal functions have been discovered. In India, the works of Dr. B. Mohan stand conspicuous in this direction. Recently Mr. P. C. Mittal, a research scholar at the University of Lucknow, has undertaken the task of correlating a self-reciprocal function with Laplace's integral. He has so far succeeded in giving two theorems and his investigations will appear in the *Journal of the Indian Mathematical Society.*

SYMPOSIUM ON THE ORIGIN OF THE SOLAR SYSTEM

Chairman: PROF. A. C. BANERJI, (*Allahabad*)

D. N. MOGHE, *R. R. College, Bombay*, opened the discussion.

The first attempt to give a rational theory of the origin of the solar system was by Kant. He thought that the planets were born out of a primeval nebula. But his was a purely non-mathematical theory. Laplace built up Kant's hypothesis on mathematical lines. It was supposed that the primeval nebula shed a ring of matter every time it contracted due to rotational motion. Every such ring condensed into a planet. It has been shown by Jeffreys that such a ring instead of condensing into a single mass would break up into small bodies like the asteroids. Moreover, the angular momentum possessed by the Sun is 1/60 of the total angular momentum of the whole system. Laplace's theory fails to explain this fact and hence it is untenable.

Chamberlin and Moulton assumed that a passing star had a grazing encounter with the primitive Sun which released eruptive forces within it and shed a great amount of matter which formed the planets. The hypothesis put forth by Jeans was that the tidal attraction of the star was responsible for the ejection of matter, while Jeffreys supposed that there was a direct collision between the star and the Sun creating a filament which condensed into planets. The star went away along its hyperbolic path and the planets (or, the filament) was altogether under the gravitational influence of the Sun. But, in such a case, calculation showed that the angular momentum had such a trifling value that the size of the orbits could not be explained.

An advance over this was made by Lyttleton who supposed that the Sun was primarily a double star and the collision due to the near approach of an intruding star gave rise to a filament which ultimately broke up into different planets moving along different orbits. The different sizes of these orbits can be accounted for on this theory. But it has been shown by Luyten and Hoyle and Bhatnagar that the conditions for the escape of the companion and the intruder are difficult to fulfil.

In his presidential address to the Mathematics Section of the Indian Science Congress held at Madras in 1940, Prof. A. C. Banerji suggested that the Sun itself was the intruding star which had a close or grazing collision with the companion of a double star and that there was a planetary filament created between this companion and the Sun. This is, however, found to be untenable as the result of the recent development in this direction due to Dr. Spitzer and Prof. H. N. Russell.

A cursory review of all these different theories will lead us to the conclusion that the main objections against all these theories are purely of a dynamical nature. The outstanding difficulties are: (i) What is the proper source from which we can get the requisite angular momentum for the Sun and the planets? (ii) How can we account for the different sizes of the planets and their orbits? Unless a satisfactory hypothesis is put forward so as to explain these points no theory will find a firm ground.

But, quite recently, objections have been raised against the "filament theories", objections which are not dynamical but which fall in the domain of Physics. It must be a fact that matter composing the filament is at extremely high temperature so that gravitational attraction is too feeble to affect the enormous speed of the individual atoms. They would, therefore, escape to great distances at any rate beyond the attractive influence of the primitive Sun. On the contrary the process of radiation would cool the gaseous mass forming the filament effecting condensations in the filament. Laws of gas opacity can only decide which of the two will have the upper hand. Dr. Spitzer concludes that expansion being much faster than cooling, the filament is reduced to a mere envelope round the Sun which is absolutely incapable of forming the planets.

Quite recently, Lyttleton has proposed a theory on the principle of rotational instability to remove the necessary material from the star to form the planets. This is a revival of the fission theory in which the Sun is again a double star. The Sun breaks up into two bodies thus forming a triple system. He finds that such systems are frequent in nature.

We have thus seen that the encounter or the collision theories do not satisfactorily explain the formation of the Solar System. Now, two alternatives are before us: whether to propose an altogether new theory, or, whether to revive an old one with the necessary improvements? Our president, Prof. Banerji is shortly going to enlighten us with his new theory. I for myself am of the opinion that a revival of the old nebular theory with the necessary improvements will lead us to a fruitful alley. It is well known that condensations are formed at different stages along the spiral arms of a nebula. These condensations are at different distances from the central body which is the primitive Sun. The process of successive cooling and contracting will reduce the enormous size of this primitive system to the present size of the solar system. As the process of cooling is a very slow one the age of the solar system will have a very high value which will perhaps agree with the geological estimate. It will be possible on this theory to explain the different sizes of the planetary orbits. Work on these lines is still in progress and when all the necessary results are obtained a paper on this subject will be communicated for publication.

C. P. S. MENON, (*Dehra Dun*)

who followed dealt briefly with the fundamental problem of the origin of the solar system and (1) the theories of rotational instability (2) tidal

theory of Jeans (3) the collision theory, and the objections to each of these. He remarked that the theory suggested by Prof. Moghe seemed to be open to the same objection as the old theory, viz. (*a*) how the critical lentecular shape could have fallen back to the spherical shape and (*b*) how the filaments shed through the equilibrium points could condense into detached masses instead of undergoing disruption due to gravitational instability. Prof. Banerjee's theory appeared to be a useful contribution provided that the amplitudes of pulsating cepheid variables could, according to actual data, be so big that the second order terms made the series divergent.

PROF. A. C. BANERJI. (*Allahabad*)

From time to time various theories of the Origin of the Solar System have been proposed, but none of them have stood the test. The planetesimal theory of Chamberlin and Moulton and the tidal theories of Jeans and Jeffreys in which the Sun and the intruding star narrowly missed each other were discounted by Russell on the ground that the weighted average of the angular momentum per unit mass for all the planets is found to be nearly ten times greater than the amount which the Star can possess due to its orbital motion relative to the Sun. Lyttleton examined mathematically Russell's suggestion that the Sun might have been a component of a binary star whose companion had been removed by a close encounter with a passing star. Luyten and Hill criticised Lyttleton's theory on the ground that the kinetic energy required to form the planets is very large and that only 6/ of the planetary ribbon could be captured by the Sun. Luyten further pointed out that, if the intruding star was sufficiently massive to give the requisite kinetic energy, the Sun was unlikely to capture any part of the ribbon without itself being captured by the intruder. Very recently Bhatnagar has shown mathematically that at the middle of the encounter the Sun would be so near the intruding star that a collision or a close encounter between the two could hardly be avoided. From astrophysical considerations Spitzer proved that even if a planetary ribbon be formed by a close encounter or a grazing collision between two stars, it will be diffused in space without giving birth to the planets.

Ross Gunn has considered the case of a rapidly rotating star breaking up by fission possibly in the vicinity of another star. The main difficulty in this theory is that such close encounters in which one of the approaching stars is rotating with angular velocity almost equal to its critical value of fission are highly improbable. Very recently Lyttleton has also considered the question of planetary formation resulting from the break-up of a single stellar mass due to rotational instability. His theory is open to the same objection as his earlier theory of planetary formation depending on encounters.

It thus appears that no existing tidal theory can explain the origin of the Solar System. To obviate the difficulties of the tidal theories, the Sun in the present theory is supposed to be originally a part of a Cepheid variable of about nine times the Sun's mass, which oscillated with small amplitude. The

nearby passage of a star of about the Cepheid's mass increased the amplitude of the osclllations rendering it unstable. That this is possible has been mathematically shewn in this paper. Matter was consequently thrown out which condensed into the Sun and the planets. Sufficient energy for this ejection was available in the parent Cepheid. The Sun is shewn to have taken about two-fifths of the energy of the parent Cepheid. It has also been shewn that the encounter need not have been very close nor the intruding star have a very high velocity to give the requisite angular momentum to the Sun and its planets to escape from the parent Cepheid. It is to be noted that the two main difficulties of the theory, viz. of energy and angular momentum, have been met here.

According to this theory, the parent Cepheid would also have a planetary system round it, and the planetary systems in our universe would be much more frequent.

BUSINESS MEETING OF THE INDIAN MATHEMATICAL SOCIETY

1. The Society paid tributes to the memory of the late Mr. M. T. Naraniengar, and Dr. Sir Shah Mohammad Sulaiman, all the members standing.

2. The Reports of the Secretary, the Librarian and the Joint Editor were read and adopted.

It was agreed that a list of books and periodicals available in the Society's library at Poona should be printed as early as possible and made available to members.

3. Suggestions regarding the advisability of having a uniform rate of the present subscription of Rs. 10/- for all members were considered. It was resolved that there should be no change in our rates of subscription, and that life composition fees be not accepted during the period of the present war.

It was also resolved that if the subscription for each year was not paid by the 1st of March, they may be recovered by the issue of our publications or of a card by Value Payable Post.

4. It was also resolved that every application for admission as a member should be accompanied by a remittance of a year's subscription before consideration by the Committee.

5. A resolution was passed thanking the Bombay University for their generous gift of Rs. 200/- annually for 5 years and for the payment of arrears since the lapsing of the previous grant.

6. The offer of Dr. A. Narasinga Rao to institute a prize for the encouragement of research to be awarded at the biennial Conferences was accepted with thanks and the Managing Committee was asked to frame the necessary rules.

7. At the suggestion of Prof. Levi, the desirability of students who had completed a course of advanced study in one University being given facilities for work in other Universities was discussed

It was agreed that

(i) Universities be requested to arrange for research scholars to be sent to specialists and that each University should pay the expenses of its students.

(ii) that Universities and Colleges be requested, whenever possible, to arrange special courses of lectures open to the post-graduate students of all Universities.

(iii) that intimation of such lectures be sent to Heads of departments of mathematics of various Universities, at least six months before the commencement of the lectures.

GLEANINGS

ROB ROY'S PROBLEM

A column of infantry one mile long is marching forward at a constant rate of speed. An officer, mounted on horseback, starts at the rear of the column and rides forward at a uniform rate of speed which is faster than that of the infantry. He rides until he catches the head of the column and then instantly turns back and rides again to the rear of the column, reaching the rear at precisely the moment it passes the point at which the head started when the officer began his ride. How far did the officer ride? No time is allowed for his turning around to ride back.

[Ans. $1+\sqrt{2}$ miles]

" most so-called philosophers, when they venture into mathematics, sell us only *aegri somnia* (sick man's dreams) for philosophy.

CARL FRIEDRICH GAUSS.

DISCUSSION ON THE TEACHING OF MATHEMATICS IN SCHOOLS AND COLLEGES

Chairman: SIR ZIAUDDIN AHMAD.

Opening the discussion SIR ZIA-UD-DIN AHMAD (Aligarh) said that Mathematics was expanding and that in order to include the new developments it became necessary to take over more and more into the pre-university course. Thus Arithmetic could be finished before the high school stage, while Algebra up to the Binomial Theorem and the elements of solid Geometry and Trigonometry might be included in the school course. The idea of a function and the elements of the calculus should find a place in the Intermediate stage, where there must be a combination of the intuitive as well as the analytical methods of approach. In the B. A. course one had to keep in mind also the requirements of students who would specialize later in Physics, by including a study of differential equations, especially those of mathematical Physics. Of advanced studies in the M. A. classes, he said that students of one college might go over to other colleges for such lectures. All training colleges should arrange for courses and lectures for the teaching of particular subjects, and all teachers invited to attend them.

C. P. S. MENON (Dehra Dun) stressed the claims of Mechanics for a place in the various syllabuses. He was against the mixing up of the several subjects in a composite course.

S. K. ABHAYANKAR (Gwalior) said that it was necessary to follow the historic method. New knowledge must be correlated with the old by pointing out analogies between the two.

M. ZIA-UD-DIN (Lahore) pleaded for the inclusion of laboratory work in statistics for the Inter and B.A. students.

KAZI KHURSHEED AHMAD (Allahabad) said there was too much of mechanical copying by students and that the standard of students was going down. Discussions should be encouraged and something done to improve the methods of teaching.

D. D. KOSAMBI (Poona): We had to ask ourselves in each case for what purpose we were teaching and whom we were teaching. The language difficulty was partly responsible for our troubles. As an instance of muddled notions regarding Newton's third law, he instanced the explanation of some pupils that though a horse and cart pulled each other with an equal force, the cart moved forward because the horse was the stronger of the two !

G. R. GUNJIKAR (Bombay) said that the setting of question papers and the marking of scripts should receive greater consideration than had been given to them. He was for an early introduction of algebraic symbolism.

B. RAMAMURTI (Ajmer) said that the utility aspect of mathematics should not override the artistic, and that we owed a duty in the matter of educating our teachers, as important as that we owed to students.

RAM BEHARI (Delhi): The historical background should be used to make the subject more interesting. Few students knew that Algebra was derived from Alkhowarezmi, or that the credit of solving the Pellian equation should go to Bhaskara. Fully rigorous proofs could not be given at the B. A. stage, but we should tell the student the conditions under which the result fails.

A. C. BANERJI (Allahabad) complained that Mechanics was taught in a haphazard manner. Physical principles were not clearly understood and much time was spent in a mechanical solution of problems.

G. R. SETH (Delhi) wanted Statistics and Actuarial Science, in the collegiate course so as to fit the student for employment after taking his degree.

RAM DHAR MISRA (Lucknow) complained that teaching was often limited to probable questions. He wanted that teaching should keep pace with the developments in the subject.

A. NARASINGA RAO (Annamalainagar) suggested that the columns of *The Mathematics Student* might be used as a pillory for exposing mistakes in university examination papers and in books, so as to ensure a high standard in both. He regretted that in bringing out text books, experienced teachers in India were not coming forth as freely as younger men in the profession. While theoretical mathematics received fair attention, little was done in class lectures to indicate the wide range of applicability of the subject to various situations in life.

MISS V. THAKURDAS (Lahore) remarked that in the Punjab, applied mathematics was overshadowed by pure mathematics.

W. A. DHANI (Student, Aligarh) said that education must not be divorced from the bread and butter problem.

Winding up the debate, SIR ZIA-UD-DIN AHMAD surveyed the various points discussed and said that teaching should not be subordinated to examinations. He thought that a group of mathematicians might be asked to set standard question papers.

Thanking Dr. Zia-ud-din, DR. VAIDYANATHASWAMI (Madras) said there had been a full discussion of important aspects of teaching. It was necessary to decide the place and degree to which we should insist on a knowledge of subjects like statistics, vector analysis in a general university course.

MEMBERS PRESENT AT THE CONFERENCE AT ALIGARH

S. Abbas Hassan Razvi, M.A., Hyderabad (Deccan).
M. Abdulla Butt, M.A., Aligarh.
S. K. Abhyankar, M.SC., L.T., Gwalior.
Afzal Ahmad, M.A., Hyderabad (Deccan).
Akhury Vindyachal Prasad, M.SC, Chapra (Bihar).
F. C. Auluck, M.A., Lahore.
Banerji, Ekanath, M.A., Cawnpore.
Banwari Lal, M.A., Meerut.
P. B. Bhattachari, M.SC., Delhi.
K. R. Gunjikar, B.SC., M.A., Bombay.
H. L. Gupta, M.SC., Cawnpore.
Hansraj Gupta, M.A., Ph.D., Hoshiapur.
S. M. Kerawala, M.A. (Cantab.), Aligarh.
M. K. Keval Ramani, M.A., Karachi.
R. Krishnamoorti, M.A., Hyderabad.
A. M. Kureishy, M.A., Aligarh.
F. W. Levi, Dr. Phil. Nat., Calcutta.
D. J. Madan, M.SC., LL.B., Karachi.
C. P. S. Menon, M.A., M.SC. (London), Dehra-Dun.
P. C Mittal, M.A., B.A. (Hons.), Lucknow.
D. N. Moghe, M. SC., Bombay.
Mohammad Shabbar, M.SC., Aligarh.
A. Narasinga Rao, M.A., D SC., L.T., Annamalainagar.
M. N. Parischa, M SC., Lucknow.
S. S. Pillai, M SC., Trivandram.
Qazi K. Ahmad, M.SC., Allahabad.
Ram Ballabh, M SC., Lucknow
Ram Behari, M.A., Ph.D., Delhi.
Ram Dhar Misra, Ph.D., Lucknow.
V. Rangachariar, M.SC., Patna.
M. Raziudin Siddiqi, Aligarh.
Sahib Ram, M.A., Lahore.
P. N. Saxena, M.A., Lucknow.
B. R. Seth, D.SC., Delhi.
S. M. Shah, M.A., Aligarh.
P. N. Sharma, M.A., Delhi.
S. W. Shiveshawarkar, M.A. (Cantab.), B.SC., I.C.S., Delhi.
P. D. Shukla, M.A., Lucknow.
A. M. Singh, Lucknow.
C. N. Sreenivasiengar, D.SC., Bangalore.
R. S. Varma, D.SC., Lucknow.
R. Vaidyanathaswamy, M.A., Ph.D., D.SC., Madras.
Zia-ud-din Ahmad, M.A. (Cantab.), Ph.D., D.SC., Aligarh.

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RULED SURFACES WHOSE CURVED ASYMPTOTIC LINES CAN BE DETERMINED BY QUADRATURES

BY

RAM BEHARI, *University of Delhi*

Abstract. 1. Picard[1] has shown that if the generators of a ruled surface belong to a linear complex, then the curved asymptotic lines can be determined by quadratures. The determination of the asymptotic lines of some other special kinds of ruled surfaces by quadratures has also been considered by Buhl,[2] Goursat,[3] Srinivasiengar[4] and Hayashi.[5] The object of this paper is to approach the problem of finding ruled surfaces whose curved asymptotic lines can be determined by quadratures in a different way by examining certain conditions under which the Riccati's equation can be integrated.

2. Let the equations of the ruled surface be

$$x = p + lu, \quad y = q + mu, \quad z = r + nu,$$

where p, q, r; l, m, n are functions of s, the arc of the base curve.

The fundamental magnitudes of the second order L, M, N are given by $LV = [x_1, x_2, x_{11}] = [l, p' + ul', 0] = 0$, where the notation $[x_1, x_2, x_{11}]$ denotes

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_{11} & y_{11} & z_{11} \end{vmatrix}$$

$$MV = [x_1, x_2, x_{12}] = [l, p' + ul', l'] = \sum l\,(q'n' - r'm') \equiv \delta, \text{ say},$$

$$NV = [x_1, x_2, x_{22}] = [l, p' + ul', p'' + ul'']$$
$$= \sum l\,(q'r'' - r'q'') + u \sum l\,\{(m'r'' - m''r') + (q'n' - n'q'')\}$$
$$+ u^2 \sum l\,(m'n'' - n'm'') \equiv \lambda + \mu u + \nu u^2, \text{ say},$$

where $\lambda = \sum l\,(q'r'' - r'q'')$,

$$\mu = \sum l\,\{(m'r'' - m''r') + (q'n'' - n'q'')\},$$

$$\nu = \sum l\,(m'n'' - n'm'').$$

The general equation of the asymptotic lines on the surface, $Ldu^2 + 2Mdudv + Ndv^2 = 0$, reduces to $dv(2Mdu + Ndv) = 0$. Hence one system of asymptotic lines is given by $v = \text{const.}$, i.e. by the generators. The second system of asymptotic lines viz. the system of curved asymptotic lines is given by the differential equation $2Mdu + Ndv = 0$, or

$$\frac{du}{dv} + \frac{\lambda}{2\delta} + \frac{\mu}{2\delta} \cdot u + \frac{\nu}{2\delta} \cdot u^2 = 0,$$

it being assumed that $\delta \neq 0$ so that the ruled surface is non-developable.

This is a Riccati's equation and cannot be integrated in finite terms for all values of λ, μ, ν, δ, i.e., for all ruled surfaces. If, however, one particular solution of Riccati's equation is known, all others can be found by quadratures.[6]

We shall examine certain conditions under which this Riccati's equation can be integrated.

3. I. If $\nu \equiv \sum l\,(m'\,n'' - n'\,m'') = 0$, i.e., if the generators of the ruled surface remain parallel to a fixed plane, the Riccati's equation reduces to the linear equation $\dfrac{du}{dv} + \dfrac{\lambda}{2\delta} + \dfrac{\mu}{2\delta} \cdot u = 0$, and asymptotic lines can be determined by two quadratures.

Taking the fixed plane as the plane of xy, the equation of the surface can be written, without loss of generality, in the form $x = u$, $y = v + f(v)\,u$, $z = F(v)$.

Here $\delta \equiv \sum l\,(q'\,n' - r'\,m') = -f'\,F'$,

$$\mu \equiv \sum l\,\{(m'\,r'' - m''\,r') + (q'\,n'' - n'\,q'')\} = -(f'\,F'' - f''\,F'),$$

$$\lambda \equiv \sum l\,(q'\,r'' - r'\,q'') = F''.$$

$\therefore$ the equation giving the curved asymptotic lines becomes

$$\frac{du}{dv} - \frac{F''}{2f'\,F'} - \frac{1}{2f'\,F'}(f'\,F'' - f''\,F')\,u = 0,$$

whose solution is

$$u = \sqrt{\frac{F'}{f'}}\left(\int \frac{F''}{2\sqrt{f'\,F'^3}}\,dv + \text{const.}\right).$$

Two cases present themselves in which the asymptotic lines can be determined in finite form

(i) Let $f'=\text{const}=c_1$ (say), so that $f=c_1 v+c_2$.

The equations of the surface are of the form

$$x=u, \quad y=v+(c_1 v+c_2) u, \quad z=F(v).$$

All the generators meet $y-c_2x=0,\ 1+c_1x=0$.

The equation of the asymptotic lines reduces to $v=\text{const}$.

(ii) Let $c_3 f'=F'$ where c_3 is a constant, so that $F=c_3 f+c_4$.

The equations of the surface are of the form

$$x=u, \quad y=v+uf(v), \quad z=c_3 f(v)+c_4.$$

The equation of the asymptotic lines takes the form

$$u=-\frac{c_3}{2F'}+\text{const}.$$

II. If $\lambda\equiv\sum l\,(q'\,r''-r'\,q'')=0$, i.e., if the generators are perpendicular to the binormals of the directrix curve, which implies that the base curve is an asymptotic, the Riccati's equation reduces to

$$\frac{du}{dv}+\frac{\mu}{2\delta}u+\frac{\nu}{2\delta}u^2=0,$$

which is Bernoulli's equation whose solution is

$$\frac{1}{u}\cdot e^{-\int\frac{\mu}{2\delta}dv}=\int e^{-\int\frac{\mu}{2\delta}dv}\cdot\frac{\nu}{2\delta}\,dv+\text{const}.$$

The condition $\lambda=0$ is satisfied by all surfaces in whose equations any two of the three quantities $p,\ q,\ r$ are either constants or zero. The most general form of the equation of such surfaces is

$$x=k_1+u, \quad y=k_2+u\,f(v), \quad z=\phi(v)+u\,\psi(v).$$

Remarks. (*a*) A special case of such ruled surfaces which has been considered by Srinivasiengar[7] is a surface of degree n which possesses a multiple line of the $(n-1)$th order.

The equation of such a surface can be written in the parametric form $x=u,\ y=uv,\ z=f(v)+u\phi(v)$.

Here $\delta = -f'(v)$, $\lambda = 0$, $\mu = f''(v)$, $\nu = \phi''(v)$.

$\therefore$ the Riccati's equation reduces to

$$\frac{du}{dv} - \frac{f''(v)}{2f'(v)}u - \frac{\phi''(v)}{2f'(v)}u^2 = 0,$$

whose solution comes out to be

$$\frac{1}{u}\cdot[f'(v)]^{\frac{1}{2}} = \int [f'(v)]^{\frac{1}{2}}\cdot\frac{\phi''(v)}{-2f'(v)}\,dv + c,$$

and can be obtained by quadratures.

(*b*) This case also includes all ruled surfaces whose line of striction is straight.

For if this straight line be taken as the directrix and be chosen as the axis of z, we can put $p=0$, $q=0$, $r=v$, and, therefore the equations of a ruled surface which has straight line of striction are of the form

$$x = lu, \quad y = mu, \quad z = v + u\cos\theta,$$

where θ is the constant angle which the generators make with the straight line of striction. These equations can be made to represent a Helicoid or a Right Conoid.[8]

III. If $\lambda=0$ and $\nu=0$, the Riccati's equation reduces to $\frac{du}{dv} + \frac{\mu u}{2\delta} = 0$ whose solution is $u = c\cdot e^{-\int\frac{\mu}{2\delta}dv}$, so that the curved asymptotic lines can be determined by one quadrature only.

This is the case when two asymptotic lines are known.

IV. If $\lambda=0$, $\nu=0$, $\mu=0$, the Riccati's equation reduces to $\frac{du}{dv} = 0$ which gives $u = \text{const}$.

This case includes the right helicoids whose equations, we know, are of the form $x = u\cos v$, $y = u\sin v$, $z = k + cv$.

N.B. Since $D \equiv lp' = 0$, the orthogonal trajectories of the generators $v = \text{const.}$ are given by $u + \int D\,dv = \text{const.}$, i.e., by $u = \text{const.}$, which gives the known result that "the curved asymptotic lines of a right helicoid are the orthogonal trajectories of the generators."

References.

1. Annales de L'ecole Normale, (1877), p. 346; or Traite d'analyse, t. 1, p. 400.

2. Nouvelles Annales de Mathematiques, (1908), pp. 340-346; Liouville, Journal de Mathemat., (1929), pp. 48, 51-52.

3. Cours d'analyse Mathematique t. 1, p. 605.

4. Jour. Ind. Math. Soc. (1931), pp. 44-48; or Zentralblatt fur Mathematik, vol. 2, (1932), p. 286.

5. Sc. Rep. Tohoku Imp. Univ. (1917), pp. 5-6.

6. Forsyth, A Treatise on Differential Equations 5th ed., p. 192; also Bour, Journal d'ecole polytechnique (1862), p. 48.

7. loc. cit.

8. Hayashi, loc. cit., pp. 5-6.

INFINITESIMAL AUTOMORPHISMS OF THE ACTION FORM

BY

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Closely associated with the Lagrangian is the fundamentally important Pfaffian form

$$A_d = p_r dq^r - H dt$$

which may be called the *Action form.* (The repeated index stands for summation from 1 to n except when otherwise mentioned.) This form, taken to be of class $2n+1$ in the $2n+1$ variables (q, p, t) is an epitome of most of the dynamical concepts* when examined from the class-theoretic point of view, while the motion of the dynamical system associated with A_d can be regarded as the 'gradual self unfolding '† of a suitable contact transformation of A_d. The general infinitesimal contact transformations of A_d include as a subgroup the infinitesimal automorphisms of A_d transforming $p_r dq^r - H dt$ into $P_r dQ^r - H' dT$.

The following are some of the interesting questions that arise:

(i) can the infinitesimal automorphisms of A_d be derived as a special case from the solution of the general infinitesimal contact transformations of A_d?,

(ii) what is the necessary and sufficient condition for the Hamiltonian H also to admit the infinitesimal automorphisms of A_d?,

and (iii) is it possible for the transformation paths to include dynamical trajectories which are the integral curves of A_d?

The object of the present paper is to answer these questions.

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* K. Nagabhushanam, On the form $\Sigma p_r dq^r - H dt$. *Proc. Ind. Ac. Sci.* Bangalore, Vol. 1, No. 8., (1935).

† E. T. Whittaker: *Analytical Dynamics*, 3rd edn. p. 304. The infinitesimal contact transformations whose gradual unfoldment represents the motion of the dynamical system have been studied by the author as I. T. transformations in his paper on 'The Transformation Theory of Dynamics in the manifold of States and Time'. *Journal. Ind. Math. Soc.* Vol. 20, p. 238.

The general contact transformations of A_d are given in terms of the arbitrary generating function W, if A_d is changed into A_d+dW. In the present case of infinitesimal automorphisms of A_d, $dW=0$, with the result that there is no differential equation for W. Hence it is evident that the infinitesimal automorphisms of A_d should not be sought as a particular case of the general infinitesimal contact transformations of A_d.

The results of this paper show that H admits the infinitesimal automorphisms of A_d if and only if $\dfrac{\partial H}{\partial t}=0$.

Again class theoretic considerations make it clear that the paths of infinitesimal automorphisms of A_d cannot include the dynamical trajectories.

Infinitesimal automorphisms of A_d

The method suited for the present inquiry is due to L. P. Eisenhart.* The solution of his homogeneous contact transformations forms a special case of the solution here.

Let the infinitesimal automorphisms be denoted by

$$\delta q^r=\phi\xi^r\,\delta\tau,\ \delta p_r=\varphi\eta_r\,\delta\tau,\ (r=1,2,\ldots n);\ \delta t=\varphi\zeta\delta\tau,$$

where φ is arbitrary. Then

$$\delta A_d=(p_r+\varphi\eta_r\,\delta\tau)\,d\,(q^r+\varphi\xi^r\,\delta\tau)-p_r\,dq^r$$
$$\left(-H+\frac{\partial H}{\partial q^r}\,\varphi\xi^r\,\delta\tau+\frac{\partial H}{\partial p_r}\,\varphi\eta_r\,\delta\tau+\frac{\partial H}{\partial t}\,\varphi\zeta\delta\tau\right)d\,(t+\varphi\xi\delta\tau)$$
$$+H dt.$$

Assuming that $d\delta\tau=0$, and equating to zero the coefficients of $\varphi\ \delta\tau\ dq^k$, $\varphi\ \delta\tau\ dp_k$, $\varphi\ \delta\tau\ dt$, $d\varphi\delta\tau$ and neglecting small quantities of higher order, we have

$$\eta_k+p_r\frac{\partial\xi^r}{\partial q^k}-H\frac{\partial\zeta}{\partial q^k}=0 \qquad \ldots\ (1),$$

$$p_r\frac{\partial\xi^r}{\partial p_k}-H\frac{\partial\zeta}{\partial p^k}=0 \qquad \ldots\ (2),$$

$$p_r\xi^r-H\zeta=0 \qquad \ldots\ (3),$$

$$p_r\frac{\partial\xi^r}{\partial t}-H\frac{\partial\zeta}{\partial t}-\frac{\partial H}{\partial t}\zeta-\frac{\partial H}{\partial q^r}\xi^r-\frac{\partial H}{\partial p_r}\eta_r=0 \qquad \ldots\ (4).$$

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* *Contact Transformations*, Annals of Mathematics, Second Series, Vol. 30, No. 2, p. 224.

If we put $p_r \xi^r = f$, we have

$$p_r \frac{\partial \xi^r}{\partial q^k} = \frac{\partial f}{\partial q^k}; \quad \xi^k + p_r \frac{\partial \xi_r}{\partial p_k} = \frac{\partial f}{\partial p_k}; \quad p_r \frac{\partial \xi^r}{\partial t} = \frac{\partial f}{\partial t}.$$

The equations (1), (2), (3), (4) becomes

$$\eta_k = -\frac{\partial f}{\partial q^k} + H\frac{\partial \zeta}{\partial q^k} \quad \ldots \quad (1'),$$

$$\xi^k = \frac{\partial f}{\partial p_k} - H\frac{\partial \zeta}{\partial p_k} \quad \ldots \quad (2'),$$

$$f - H\zeta = 0 \quad \ldots \quad (3'),$$

so that

$$\frac{\partial f}{\partial t} - \frac{\partial H}{\partial t}\zeta - H\frac{\partial \zeta}{\partial t} = 0 \quad \ldots \quad (3'a)$$

and

$$\frac{\partial H}{\partial q^r}\xi^r + \frac{\partial H}{\partial p_r}\eta_r = 0 \quad \ldots \quad (4')$$

by virtue of $(3'a)$.

It is easily seen by direct calculation that (4′) is a consequence of (1′), (2′) and (3′). For

$$\begin{aligned}
\frac{\partial H}{\partial q^r}\xi^r + \frac{\partial H}{\partial p_r}\eta_r - \frac{\partial H}{\partial q^r}\left(\frac{\partial f}{\partial p_r} - H\frac{\partial \zeta}{\partial p_r}\right) + \left(-\frac{\partial f}{\partial q^r} + H\frac{\partial \zeta}{\partial q^r}\right) \\
= \frac{\partial H}{\partial q^r}\left(\frac{\partial f}{\partial p_r} - \frac{\partial H}{\partial p_r}\zeta - H\frac{\partial \zeta}{\partial p_r}\right) \\
+ \frac{\partial H}{\partial p_r}\left(-\frac{\partial f}{\partial q^r} + \frac{\partial H}{\partial q^r}\zeta + H\frac{\partial \zeta}{\partial q^r}\right) \\
= 0, \text{ since } \left.\begin{aligned} \frac{\partial}{\partial q^r}(f - H\zeta) = 0 \\ \text{and } \frac{\partial}{\partial p_r}(f - H\zeta) = 0 \end{aligned}\right\} \text{by } (3').
\end{aligned}$$

Hence we have

THEOREM 1.—*If f and ζ are two functions satisfying the relation $f - H\zeta = 0$, the equations (1′) and (2′) give the most general solution of the infinitesimal automorphisms of A_d.*

If f is homogeneous of degree one in the p's, and ζ an absolute constant, the solutions of ξ^k, η_k give the homogeneous contact transformation of Eisenhart.

Conditions under which H admits the infinitesimal automorphisms of A_d

$$\delta H = \frac{\partial H}{\partial q^r}\varphi\xi^r\delta\tau + \frac{\partial H}{\partial p_r}\varphi\eta_r\delta\tau + \frac{\partial H}{\partial t}\varphi\zeta\delta\tau$$

$$\varphi\frac{\partial H}{\partial t}\zeta\delta\tau \qquad \text{by } (4')$$

Hence if $\delta H = 0$, then $\frac{\partial H}{\partial t} = 0$, since $\varphi\zeta\delta\tau$ is arbitrary. Also $\delta H = 0$ if $\frac{\partial H}{\partial t} = 0$. Thus we have

THEOREM 2.—*The condition that H is explicitly independent of t is both necessary and sufficient for H to admit the infinitesimal automorphisms of A_d.*

We shall now establish the following result:

THEOREM 3. *The paths of infinitesimal automorphisms of A_d cannot include the dynamical trajectories.*

In this last section, A_d is written as $X_i dx^i$, where i runs through the indices $1, 2, \cdots 2n+1$, and the class of $X_i dx^i$, viz., the rank of the matrix $|a_{ik};\ X_k|$ where $\left(a_{ik} = \frac{\partial X_i}{\partial x^k} - \frac{\partial X_k}{\partial x^i}\right)$, is $2n+1$. Let the infinitesimal automorphisms of A_d be now denoted by

$$\delta x^i = \phi\ \sigma^i\ \delta\tau, \qquad i = 1, 2 \cdots 2n+1.$$

$$\delta A_d = \left(X_i + \frac{\partial X_i}{\partial x^k}\varphi\sigma^k\delta\tau\right) d(x^i + \varphi\sigma^i\delta\tau) - X_i\, dx^i.$$

Equating to zero, the co-efficients of $\varphi dx^i\delta\tau$, and $d\varphi\delta\tau$, we obtain

$$\left.\begin{aligned} &\frac{\partial X_i}{\partial x^k}\sigma^k + \frac{\partial \sigma^k}{\partial x^i}X_k = 0, \qquad (i = 1, 2 \cdots 2n+1) \\ \text{and} \qquad &X_i\,\sigma^i = 0 \end{aligned}\right\} \qquad \dots (5)$$

If the trajectories are to be included in the paths of the transformation, σ^i must satisfy the equations (5) whenever $a_{ik}\,\sigma^k = 0$.

Hence $X_k\sigma^k = 0$ is to be a consequence of the set $a_{ik}\sigma^k = 0$, with the result that the rank of the matrix $|a_{ik};\ X_k|$ is to be the same as that of $|a_{ik}|$ viz. $2n$ or less. Thus the class of A_d will have to be $\leqslant 2n$. As this is impossible, the trajectories cannot be included in the paths of the transformation.

ON THE VOLUME OF A PRISMOID IN N—SPACE AND SOME PROBLEMS IN CONTINUOUS PROBABILITY.

BY

C. RADHAKRISHNA RAO.

Let $x_1, x_2 \cdots\cdots x_n$ be n independent variables. An ordered set $(x_1, x_2 \cdots\cdots\cdots x_n)$ can be said to represent a point in n dimensional space. A linear equation of these variables such as $l_1 x_1 + l_2 x_2 + \cdots\cdots = k$ represents a hyperplane in n dimensions.

The hyperplanes
$$\begin{array}{ll} x_1 = a_1, & x_1 = b_1, \\ x_2 = a_2, & x_2 = b_2, \\ \cdots\cdots & \cdots\cdots \\ x_n = a_n, & x_n = b_n, \end{array}$$
enclose a certain space D called the Prismoid.

§1. *To find the volume either exterior or interior to the prismoid cut off by the hyperplane*

$$l_1 x_1 + l_2 x_2 \cdots\cdots = k.$$

Starting with a simple case let it be required to find the volume enclosed by

$$\left.\begin{array}{ll} (x_1 > c_1,) & 0 \leqslant x_{i+1} \leqslant c_{i+1} \\ (x_2 > c_2,) & \cdots\cdots\cdots \\ (\cdots\cdots) & 0 \leqslant x_n \leqslant c_n \\ (x_i > c_i,) & 0 \leqslant S(x_n) \leqslant P \end{array}\right\} \qquad (1{\cdot}1)$$

where $S(x_n)$ means $x_1 + x_2 \cdots\cdots + x_n$.

Obviously $P > c_1 + c_2 \cdots\cdots\cdots c_i$.

This volume is the same as that enclosed by

$$\begin{array}{ll} (y_1 > 0) & 0 \leqslant y_{i+1} \leqslant c_{i+1} \\ (y_2 > 0) & 0 \leqslant y_n \leqslant c_n \\ (\cdots\cdots) & 0 \leqslant S(y_n) \leqslant P - (c_1 + c_2 \cdots\cdots + c_i) = Q \\ (y_i > 0) & \end{array}$$

in a different space, got by the substitutions

$$x_1 - c_1 = y_1, \qquad x_s - c_s = y_s$$
$$x_2 - c_2 = y_2, \qquad x_{s+1} = y_{s+1}$$
$$\cdots\cdots \qquad x_n = y_n$$

The volume of the region bounded by the coordinate planes and the plane $S(y_n) = Q$ is given* by $\frac{Q^n}{n!}$ (A)

Let $Q > c_s$ $\quad s = h, u, v, \cdots\cdots$

Then the above expression includes the volume above certain faces

$$y_s = c_s \qquad s = h, u, v \cdots\cdots$$

of the domain D. The sum of the different volumes above these faces is

$$\frac{(Q - c_s)^n}{n!} + \frac{(Q - c_u)^n}{n!}, \cdots \tag{B}$$

where $Q - c_r$ is greater than zero.

In case $Q > c_h + c_r$, the volume exterior to both planes $y_h = c_h$, $y_r = c_r$ has been included in (A);

This volume has been already subtracted twice, once in $\frac{(Q - c_h)^n}{n!}$ and once in $\frac{(Q - c_r)^n}{n!}$. So the volume $\frac{(Q - c_h - c_r)^n}{n!}$ above the two faces and enclosed by the above plane has to be added to (A) – (B).

Let the sum of similar terms be

$$\frac{(Q - c_u - c_r)^n}{n!} + \frac{(Q - c_v - c_t)^n}{n!} + \cdots \tag{C}$$

the bracket in each case being positive.

Similarly we can form the sums D, E, F .... and the volume is easily seen to be

$$A - B + C - D \cdots\cdots \tag{1·2}$$

The result for conditions (1·1) is obtained by writing

$$Q = P - (c_1 + c_2 \cdots + c_s)$$

* The value of the Dirichlet's integral.

If the conditions are $0 \leqslant x_r \leqslant c_r$ $(r=1 \cdot 2 \cdot \cdot \cdot \cdot \cdot n)$ then clearly the result becomes

$$\frac{1}{n!}\{P^n - S(P-c_s)^n + S(P-c_t-c_u)^n - \cdot \cdot \cdot \cdot \cdot\}$$

$$P-c_s > 0,\ (P-c_t-c_u) > 0, \text{ etc.} \tag{1·3}$$

The volume of the lower portion of the domain D formed by

$$x_r = a_r,\ x_r = b_r,\ (r=1,\ 2 \cdot \cdot \cdot n)$$

cut off by the hyperplane is given by writing

$$l_1 x_1 + l_2 x_2 \cdot \cdot \cdot = k$$

$$P = k - S\,(a_r l_r); \qquad c_r = l_r\,(b_r - a_r)$$

in (1·3) and multiplying it by $(l_1 l_2 \cdot \cdot l_n)^{-1}$, the result being obtained after suitable substitutions $l_r(x_r - a_r) = y_r$.

Incidentally, this gives the proportionate probability of the occurrence of values less than a certain magnitude for the mean of a number of observations each drawn from rectangular distributions with a finite range.

If all the c's are equal the results (1·2) and (1·3) become

$$(P-ic)^n - {}_{n-1}C_1\,(P-\overline{i+1}\,c)^n + {}_{n-i}C_2\,(P-\overline{i+2}\,c)^n$$

$$\cdot \cdot \cdot (-1)^k\,{}_{n-i}C_k\,(P-\overline{i+k}\,c)^n \tag{1·22}$$

where k is such that

$$(i+k)c \leqslant P < (i+k+l)c$$

and $\frac{1}{n!}\{P^n - {}_nC_1\,(P-c)^n + {}_nC_2\,(P-2c)^n \cdot \cdot \cdot (-1)^r\,{}_nC_r\,(P-rc)^n\}$ (1·33)

r is such that $rc \leqslant p < (r+1)c$

§2. On the circumference of a circle of unit length, n arcs of lengths $h_1, h_2 \cdot \cdot \cdot h_n$ are marked off in a random order. What are the probabilities that

1. there are r gaps;
2. there are no gaps;
3. the intervals do not overlap by a distance more than c.

W. L. Stevens* gave the result for 1 and 2 when $h_1 = h_2 \cdots$ as a deduction from a general theorem.

Let the intervals be $(0, h_1)$, $(x_1, x_1 + h_2)$, $(x_2, x_2 + h_3)$, $\cdots$ $(x_{n-1}, x_{n-1} + h_n)$ where $0 < x_1 < \cdots x_{n-1}$. If there be r gaps after the intervals of length h_{r+1}, h_{s+1} etc. but nowhere else we have

$$\begin{array}{ll} 0 < x_1 < h_1 & x_{r+1} - x_r > h_{r+1} \\ 0 < x_2 - x_1 < h_2 & x_{s+1} - x_s > h_{s+1} \\ \cdots\cdots\cdots & \cdots\cdots\cdots \end{array}$$

and finally $0 < 1 - x_{n-1} < h_n$ since there is no gap after the interval h_n

The inequalities become after substitutions $x_1 = y_1$, $x_2 - x_1 = y_2$, etc.

$$\left.\begin{array}{ll} 1 - h_n < S(y_{n-1}) \leqslant 1 & \\ 0 \leqslant y_1 < h_1 & y_{r+1} > h_{s+1} \\ \cdots\cdots & \cdots\cdots \\ 0 \leqslant y_{k+1} < h_{k+1} & y_{u+1} > h_{u+1} \end{array}\right\} \qquad (b)$$

To find the probability for this specified arrangement we have to find the volume enclosed by the above hyperplanes viz. the boundaries of $S(y_{n-1})$ in the domain defined by (b) and divide it by the total volume of the domain of freedom of $y_1\ y_2\ .\ .\ .\ y_{n-1}$ which is equal to 1.

Employing (1·2) we get

$$k = \frac{1}{(n-1)!}\ \{(P^{n-1} - S(P - h_i)^{n-1} + S(P - h_s - h_t)^{n-1}\ .\ .\ .\ )\}$$

$$= \frac{1}{(n-1)!}\ \{(P - h_n)^{n-1} - S(P - h_n - h_t)^{n-1}\ .\ .\ .\ .\ .\ .\ .\ .\ )\}$$

where $P = 1 - h_{r+1}\ .\ .\ .\ .\ .\ .\ .\ .\ - h_{u+1}$

Hence $K = \frac{1}{(n-1)!}\ \{(P^{n-1} - S(P - h_1)^{n-1} +\ .\ .\ .\ .\ .\ .\ .\ .\ )\}$

where h_n is included in the summations, a symmetrical result which does not depend on the origin chosen.

Since there are $(n-1)!$ arrangements of these intervals round the circle we get the probability for the specified arrangement

$$P_{r \ldots u} = (n-1)!\ k$$

where $P_{r \ldots u}$ denotes the probability for gaps after $r+1$, $\cdots$ $(u+1)$th intervals.

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* Annals of Eugenics 1939.

The compound probability for any r gaps is given by

$$S(P_{r,\dots u}) \tag{2·1}$$

The summation consisting of ${}_nC_r$ terms.

When there are no gaps the inequalities satisfy (1·3) and hence by similar reasoning we get

$$P_0 = 1 - S(1-h_r)^{n-1} + S(1-h_r-h_s)^{n-1} \cdot\cdot\cdot\cdot \tag{2·2}$$

In particular if $h_1 = h_2 \cdot\cdot\cdot\cdot = x$ the results become

$$P_r = {}_nC_r \{(1-rx)^{n-1} - {}_{n-r}C_1 (1-x)^{n-1} + \cdot\cdot\cdot\} \tag{2·3}$$

$$P_0 = \{1 - {}_nC_1 (1-x)^{n-1} + {}_nC_2 (1-2x)^{n-1} \cdot\cdot\cdot\cdot\cdot\cdot (-1)^k {}_nC_k (1-kx)^{n-1}\}$$

$$kx \leqslant 1 < (k+1)\, x$$

The mean number of gaps is easily seen* to be $n(1-x)^{n-1}$

To solve 3 we have the following inequalities

$$(x_1 > h_1 - c) \qquad\qquad x_{n-1} - x_{n-2} > h_{n-1} - c$$

$$(x_2 - x_1 > h_2 - c) \qquad\qquad 1 - x_n + c > h_n$$

$$\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot$$

By the usual substitution we get

$$y_r > k_r, \qquad\qquad (r = 1, \cdot\cdot\cdot\cdot\cdot\cdot\cdot\, n-1),$$

and $k_r = h_r - c$, and $S(y_r) \leqslant 1 - k_n$

This leads us to find the volume exterior to be prismoid bounded by the hyperplane $S(y_r) = 1 - k_n$

$$= \frac{(1 - k_n - k_2 \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot - k_1)^{n-1}}{(n-1)\,!}$$

$$= \frac{(1 - h_1 - h_2 \cdot\cdot\cdot\cdot\cdot\cdot - h_n + nc)^{n-1}}{(n-1)\,!}$$ provided the expression within the brackets is positive.

So the probability is given by $\{1 + nc - S\,(h_r)\}^{n-1}$

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* Discussed in the end.

If in addition we have no gaps the inequalities become

$$h_1 > x_1 > h_1 - c$$
$$h_2 > x_2 - x_1 > h_2 - c$$
$$\cdots\cdots\cdots\cdots$$

and with the boundary conditions $h_n > 1 - x_{n-1} > h_n - c$.

This again reduces to the above case on making the substitutions

$$y_1 = h_1 - x_1$$
$$y_2 = h_2 - x_2 + x_1$$
$$\cdots\cdots\cdots$$

The boundary condition becomes

$$h_n > 1 + S(y_r) - S(h_r) > h_n - c$$

or

$$-1 + S(h_r) > S(y_r) > -1 - c + S(h_r)$$

h_n included $\qquad$ h_n included

and the probability can be easily given by writing $S(h_r) - 1 - c$ and $S(h_r) - 1$ for P in (1·3) and subtracting one from the other.

In the case of a straight line the boundary condition changes. For instance, if segments of lengths, h_1, $h_2 \cdots h_n$ are marked off at random on a line of length 1 what is the probability that there are no gaps, and yet they do not overlap by a length c?

$$\underline{\text{O}\ h_1\ |\qquad\qquad\qquad|\ h_n\ \text{A}}$$
$$x_1 \qquad\qquad\qquad x_{n-1}$$

$$h_1 - c \leqslant x_1 \leqslant h_1$$
$$h_2 - c \leqslant x_2 - x_1 \leqslant h_2$$
$$\cdots\cdots\cdots\cdots$$
$$h_{n-2} - c \leqslant (x_{n-2} - x_{n-3}) \leqslant h_{n-3}$$

and $\quad 1 - x_{n-2} - h_{n-1} > h_n - c$

$$h_n + h_{n-1} > 1 - x_{n-2} > h_n + h_{n-1} - c$$

This again leads to the one of the sets of inequalities and the corresponding probability can be written. With similar considerations many problems in continuous probability can be solved.

3. **The distribution of the number of gaps**

Putting ${}_nC_r\,(1-rx)^{n-1}=x_r$ we have if P_m denotes the probability for m gaps,

$$P_0=1-x_1+x_2-x_3\cdots\cdots(-1)^k x_k$$

$$P_1=x_1-2x_2+3x_3\cdots\cdots(-1)^{k-1}\,kx_k$$

$$P_2=x_2-{}_3C_2\,x_3\cdots\cdots\cdots(-1)^{k-2}\,{}_kC_2\,x_k$$

$$P_k=\cdots\cdots\cdots\cdots\cdots+x_k$$

The moments of this distribution are

$$\mu_1'=x_1;\ \mu_2'=x_1+2x_2;\ \mu_3'=x_1+3!\,(x_2+x_3)$$

$$\mu_4'=x_1+14x_2+36x_3+4!\,x_4$$

μ's can also be written in a similar manner although the general expression is complicated.

$$\mu_1=0;\ \mu_2=x_1+2x_2-x_1^2=x_1\,(1-x_1)+2x_2$$

The distribution is skew, except when $\mu_3=0$;

$$\text{i.e., } x_1+3!\,(x_2+x_3)+2x_1^3=3x_1\,(x_1+2x_2)$$

HOWLERS

Algebra is the wife of Euclid.

A triangle is a straight line bent twice so that the two ends meet.

THE EVALUATION OF CERTAIN DETERMINANTS

BY

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1. In this paper I evaluate certain determinants whose elements are trigonometric functions.

Let $f(x)$ denote the function

$$\sum_{l=1}^{r} a_l \exp. (m_l x)$$

Then, if we denote the determinant of the nth order whose element in the ith row and jth column is $a_{i,j}$ by $D_n[a_{i,j}]$, we have

THEOREM 1.

$$D_n[f(\theta_i+\varphi_j)]=0 \text{ if } n>r.$$

Put exp. $(m_k\,\theta_i)=x_{k,i}$ and exp. $(m_k\,\varphi_i)=y_{k,i}$. Then

$$D_n\,[f(\theta_i+\varphi_j)]=D_n\Big[\sum_{l=1}^{r} a_l x_{l,i}\, y_{l,j}\Big]$$

$$=\prod_{i=1}^{n}\Big\{\sum_{j=1}^{n}\varepsilon_j\Big(\sum_{l=1}^{r} a_l x_{l,i}\, y_{l,j}\Big)\Big\},$$

where ε_j are the usual symbols (alternating units) occurring in the theory of determinants, satisfying the properties

$$\Pi\,\varepsilon_j=1,\ \varepsilon_i\,\varepsilon_j=-\varepsilon_j\,\varepsilon_i,\ \varepsilon_i^2=0.$$

Therefore

$$D_n[f(\theta_i+\varphi_j)]=\prod_{i=1}^{n}\ \sum_{l=1}^{r} a_l x_{l,i}\, A_l,$$

where A_l is the alternating number $\sum_{j=1}^{n}\varepsilon_j\, y_{l,j}$.

On multiplying out the above product we see that each term has n alternating numbers A_l as factors. If $n>r$, since there are only r distinct alternating numbers A_l, there will be repeated factors in each term. Therefore from the fact that the square of an alternating number is zero, it follows that all the terms vanish. Thus we get the required result.

COROLLARY*. $D_n[\sin(\theta_i+\varphi_j)] = D_n[\cos(\theta_i+\varphi_j)] = 0$ if $n \geqslant 3$.

2. We require the following lemmas:

LEMMA 1.

$$D_n\left[\frac{1}{x_i+y_j}\right] = \prod_{\substack{i,j=1\\(i>j)}}^{n} (x_i-x_j)(y_i-y_j) \prod_{i,j=1}^{n} 1/(x_i+y_j).$$

PROOF: $D_n\left[\frac{1}{x_i+y_j}\right]$ vanishes whenever $x_i = x_j$, $y_i = y_j$ $(i \neq j)$.

Therefore $\Pi\,(x_i-x_j)(y_i-y_j)$ is a factor of $D_n\left[\frac{1}{x_i+y_j}\right]$.

But when the determinant $D_n\left[\frac{1}{x_i+y_j}\right]$ is expanded, and the terms put on a common denominator which is simply the product of all the denominators of the elements of the determinant, the numerator will be a homogeneous expression of degree $n(n-1)$ in $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$.

It follows that

$$D_n\left[\frac{1}{x_i+y_j}\right] = k \prod_{\substack{i,j=1\\(i>j)}}^{n} (x_i-x_j)(y_i-y_j) \div \prod_{i,j=1}^{n} (x_i+y_j),$$

where k is some constant. Comparing the co-efficients of $x_2x_3^2 \cdots x_n^{n-1}y_2y_3^2 \cdots y_n^{n-1}$ in the numerators on the two sides we see that $k=1$. Thus the lemma is proved.

LEMMA 2. $D_n\left[\frac{x_i-y_j}{x_i+y_j}\right]$

$$= 2^{n-1}(x_1x_2\cdots x_n + (-1)^n y_1y_2\cdots y_n) \prod_{\substack{i,j=1\\(i>j)}}^{n} (x_i-x_j)(y_i-y_j) \times \prod_{i,j=1}^{n} \frac{1}{x_i+y_j}.$$

PROOF: It is easily seen that

$$D_n\left[\frac{x_i-y_j}{x_i+y_j}\right] = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \frac{2y_1}{x_1+y_1} & \frac{2y_2}{x_1+y_2} & \cdots & \frac{2y_n}{x_1+y_n} & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \frac{2y_1}{x_n+y_1} & \frac{2y_2}{x_n+y_2} & \cdots & \frac{2y_n}{x_n+y_n} & 1 \end{vmatrix}$$

* This may also be proved by making the substitutions $\tan\theta_i = x_i$ and $\tan\varphi_i = y_i$ and noting that $D_n[x_i+y_j] = 0$.

$$= 2^{n-1} y_1 y_2 \cdots y_n \begin{vmatrix} \frac{1}{y_1} & \frac{1}{y_2} & \cdots & \frac{1}{y_n} & 2 \\ \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \cdots & \frac{1}{x_1+y_n} & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \frac{1}{x_n+y_1} & \frac{1}{x_n+y_2} & \cdots & \frac{1}{x_n+y_n} & 1 \end{vmatrix} = 2^{n-1} y_1 y_2 \cdots y_n D', \text{ (say)}.$$

$$\text{Now } D' = \begin{vmatrix} \frac{1}{y_1} & \frac{1}{y_2} & \cdots & \frac{1}{y_n} & 1 \\ \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \cdots & \frac{1}{x_1+y_n} & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \frac{1}{x_n+y_1} & \frac{1}{x_n+y_2} & \cdots & \frac{1}{x_n+y_n} & 1 \end{vmatrix} + (-1)^n D_n\left[\frac{1}{x_i+y_j}\right]$$

But by lemma 1, $D_n\left[\frac{1}{x_i+y_j}\right] = \Pi\ (x_i - x_j)(y_i - y_j)\ \Pi\ \frac{1}{(x_i+y_j)}$

In a similar way we can prove that

$$\begin{vmatrix} \frac{1}{y_1} & \frac{1}{y_2} & \cdots & \frac{1}{y_n} & 1 \\ \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \cdots & \frac{1}{x_1+y_n} & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \frac{1}{x_n+y_1} & \frac{1}{x_n+y_2} & \cdots & \frac{1}{x_n+y_n} & 1 \end{vmatrix} = \frac{x_1 x_2 \cdots x_n}{y_1 y_2 \cdots y_n} \prod_{\substack{i,j=1 \\ i>j}}^{n} (x_i - x_j)(y_i - y_j) \times \prod_{i,j=1}^{n} \frac{1}{x_i + y_j}.$$

Thus we get $\quad D_n\left[\frac{x_i - y_j}{x_i + y_j}\right] =$

$$2^{n-1}(x_1 x_2 \cdots x_n + (-)^n y_1 y_2 \cdots y_n) \prod^{n} (x_i - x_j)(y_i - y_j) \times \prod_{i,j=1}^{n} \frac{1}{(x_i + y_j)}.$$

THEOREM 2.

$$D_n\,[\operatorname{cosec}\,(\theta_i + \varphi_j)] = \prod_{\substack{i,j=1 \\ (i>j)}}^{n} \sin\,(\theta_i - \theta_j)\,\sin\,(\varphi_i - \varphi_j) \prod_{i,j=1}^{n} \operatorname{cosec}\,(\theta_i + \varphi_j).$$

Put $\tan\theta_i = x_i$ and $\tan\varphi_i = y_i$.

Then $D_n\,[\operatorname{cosec}(\theta_i+\varphi_j)] = D_n\left[\dfrac{1}{x_i+y_j}\right]\prod_{i=1}^{n}\sqrt{(1+x_i^2)(1+y_i^2)}$.

$$= \prod_{\substack{i,j=1\\(i>j)}}^{n}(x_i-x_j)(y_i-y_j)\prod_{i,j=1}^{n}\frac{1}{x_i+y_j}\prod_{i=1}^{n}\sqrt{(1+x_i^2)(1+y_i^2)}$$

by lemma 1.

$$= \prod_{\substack{i,j=1\\i>j}}^{n}\frac{(x_i-x_j)}{\sqrt{(1+x_i^2)(1+x_j^2)}}\cdot\frac{(y_i-y_j)}{\sqrt{(1+y_i^2)(1+y_j^2)}}\prod_{i,j=1}^{n}\frac{\sqrt{(1+x_i^2)(1+y_i^2)}}{x_i+y_j}.$$

Substituting the values of x_i and y_i we get the required result.

COROLLARY. Putting $\tfrac{1}{2}\pi+\theta_i$ instead of θ_i we get

$$D_n\,[\sec(\theta_i+\varphi_j)] = \prod_{\substack{i,j=1\\(i>j)}}^{n}\sin(\theta_i-\theta_j)\sin(\varphi_i-\varphi_j)\prod_{i,j=1}^{n}\sec(\theta_i+\varphi_j).$$

THEOREM 3.

$$D_n\left[\frac{\sin(\theta_i-\varphi_j)}{\sin(\theta_i+\varphi_j)}\right] = \frac{2^{n-1}\left(\prod_{i=1}^{n}\tan\theta_i+(-1)^n\prod_{i=1}^{n}\tan\phi_i\right)}{\sec\theta_1\sec\theta_2\cdots\sec\theta_n\cdot\sec\varphi_1\sec\varphi_2\cdots\sec\varphi_n}.$$

$$\times\prod_{\substack{i,j=1\\(i>j)}}^{n}\sin(\theta_i-\theta_j)\sin(\varphi_i-\varphi_j)\prod_{i,j=1}^{n}\operatorname{cosec}(\theta_i+\varphi_j)$$

This can be proved like theorem 2 with the help of lemma 2

Replacing θ_i by $(\tfrac{1}{2}\pi+\theta_i)$, θ_i by $\pi/4+\theta_i$ and φ_i by $(\pi/4+\varphi_i)$ and θ_i by $\pi/4+\theta_i$ and φ_i by $(\varphi_i-\pi/4)$ we can get the values of

$D_n\left[\dfrac{\cos(\theta_i-\varphi_j)}{\cos(\theta_i+\varphi_j)}\right]$, $D_n\left[\dfrac{\sin(\theta_i-\varphi_j)}{\cos(\theta_i+\varphi_j)}\right]$ and $D_n\left[\dfrac{\cos(\theta_i-\varphi_j)}{\sin(\theta_i+\varphi_j)}\right]$ respectively

THEOREM 4. $D_n[\tan(\theta_i+\varphi_j)]$

$$=(-1)^{n/2}\cos(\theta_1+\cdots+\theta_n+\varphi_1+\cdots+\varphi_n)\prod_{\substack{i,j=1\\(i>j)}}^{n}\sin(\theta_i-\theta_j)\sin(\varphi_i-\varphi_j)$$

$$\times\prod_{i,j=1}^{n}\sec(\theta_i+\varphi_j),\qquad\text{if } n \text{ is even}$$

$$=(-1)^{\frac{n-1}{2}}\sin(\theta_1+\cdots+\theta_n+\varphi_1+\cdots+\varphi_n)\prod_{\substack{i,j=1\\(i>j)}}^{n}\sin(\theta_i-\theta_j)\sin(\varphi_i-\varphi_j)$$

$$\times\prod_{i,j=1}^{n}\sec(\theta_i+\varphi_j),\qquad\text{if } n \text{ is odd.}$$

By making the substitutions $\exp.(2i\theta_k) = x_k$, and $\exp.(-2i\varphi_k) = y_k$ we see that $D_n[\tan(\theta_i + \varphi_j)] = \frac{1}{i^n} D_n\left[\frac{x_i - y_j}{x_i + y_j}\right]$,

and by using lemma 2 we can get the required result.

By changing θ_i to $\frac{1}{2}\pi + \theta_i$ we can evaluate $D_n[\cot(\theta_i + \varphi_j)]$.

3. Various other determinants can also be evaluated by suitable substitutions in lemmas (1) and (2). Thus, for instance, results similar to those in the above theorems can be obtained where instead of circular functions we have hyperbolic functions.

GLEANINGS

Another characteristic calls for mention here, as several writers and artists (some from Hollywood) have asked that it be treated—the sex life of the great mathematicians. In particular these enquirers wish to know how many of the great mathematicians have been perverts—a somewhat indelicate question, possibly, but legitimate enough to merit a serious answer in these times of preoccupation with such topics. None. Some lived celibate lives, usually on account of economic disabilities, but the majority were happily married and brought up their children in a civilized, intelligent manner. The children, it may be noted in passing, were often gifted far above the average. A few of the great mathematicians of bygone centuries kept mistresses when such was the fashionable custom of their times. The only mathematician discussed here whose life might offer something of interest to a Freudian is Pascal.

E. T. BELL in *Men of Mathematics*.

"What say you," pursued he (Socrates) "upon the Sun's return after winter to revisit us; and that as the fruits of one season wither and decay, he ripens new ones to succeed them? That having rendered man this service, he retires, lest he should incommode him by excess of heat; and then after having removed to a certain point, which he could not pass without putting us in danger of perishing with cold, that he returns in the same track to resume his place in those parts of the heavens where his presence is most beneficial to us? And because we could neither support the cold nor heat, if we were to pass in an instant from the one to the other, do you not admire, that whilst this star approaches and removes so slowly, the two extremities arrive by almost insensible degrees? Is it possible not to discover, in this disposition of the seasons of the year, a providence and goodness, not only attentive to our necessities, but even our delights and enjoyments?"

From ROLLIN'S *Universal History*

THE MAXIMUM TERM OF AN ENTIRE SERIES

BY

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§1. Let $f(z)=\sum_{0}^{\infty} a_n z^n$ be an integral function; $\mu(r)$ the maximum term for $|z|=r$ and $\nu(r)$ its rank. It is known[1] that if $f(z)$ be of finite order $\rho>0$

$$\liminf_{r\to\infty} \frac{\nu(r)}{\log \mu(r)} \leqslant \rho \leqslant \limsup_{r\to\infty} \frac{\nu(r)}{\log \mu(r)} \qquad \dots \quad (1)$$

We prove here a result which includes (1)

THEOREM 1. *If $f(z)$ be of order ρ, $0<\rho<\infty$ then*[2]

$$\liminf_{r\to\infty} \frac{\nu(r)}{\log \mu(r)} \leqslant \lambda \leqslant \rho \leqslant \limsup_{r\to\infty} \frac{\nu(r)}{\log \mu(r)} \qquad \dots \quad (2)$$

where $\lambda = \liminf_{r\to\infty} \{\log\log M(r)/\log r.\}$

Corollary: *A necessary and sufficient condition that $f(z)$ be of order ρ and of regular growth is that $\nu(r)/\log \mu(r) \to \rho$, as $r\to\infty$.*

§2. We require a lemma.

Let (i) $\chi(x)$ *be a real function, positive and integrable* L *in any interval* (Δ, r) *where* $\Delta>0$;

(ii) $\chi(r) \geqslant 1$ *for all large* r;

(iii) $\limsup_{r\to\infty} \frac{\log \chi(r)}{\log r} = A$; $\liminf_{r\to\infty} \frac{\log \chi(r)}{\log r} = B$;

(iv) $I(r) = \int_{\Delta}^{r} \frac{\chi(x)dx}{x}$;

then

$$\liminf_{r\to\infty} \frac{\chi(r)}{I(r)} \leqslant B \leqslant A \leqslant \limsup_{r\to\infty} \frac{\chi(r)}{I(r)} \qquad \dots \quad (3)$$

The proof is similar to that of the lemma of my paper[3] and is omitted.

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[1] G. Polya and G. Szego: Aufgaben und Lehrsatze aus der Analysis, II Ex. 59—60 (P 9).

[2] $M(r)$, $n(r)$, $N(r, a)$, $\overset{+}{\log} A$ have their usual meanings. See E. C. Titchmarsh Theory of Functions (1939) ch. 8.

[3] Note on a Theorem of Polya, *J. I. M. S.*, 5 (1941) 189—191.

§3. PROOF OF THEOREM 1.

We have
$$\log \mu(r) = A_1 + \int_{\Delta}^{r} \frac{\nu(t)}{t} dt$$

Putting $X(r) = \nu(r)$ in the lemma[1] and noting that[2]

$$\liminf_{r\to\infty} \frac{\log \nu(r)}{\log r} = \lambda; \quad \limsup_{r\to\infty} \frac{\log \nu(r)}{\log r} = \rho,$$

we obtain

$$\liminf_{r\to\infty} \frac{\nu(r)}{\log \mu(r)} \leqslant \lambda \leqslant \rho \leqslant \limsup_{r\to\infty} \frac{\nu(r)}{\log \mu(r)}$$

§4. Let

$$\lambda_1 = \liminf_{r\to\infty} \frac{\log^+ n(r)}{\log r}\ ; \ \rho_1 = \limsup_{r\to\infty} \frac{\log^+ n(r)}{\log r}$$

If $n(r) > 0$ for some value of r then $n(r) \geqslant 1$ for all large r and we have from the lemma

$$\liminf_{r\to\infty} \frac{n(r)}{N(r,0)} \leqslant \lambda_1 \leqslant \rho_1 \leqslant \limsup_{r\to\infty} \frac{n(r)}{N(r,0)} \quad \ldots \quad (4)$$

If $n(r) = 0$ for all values of r then $\lambda_1 = \rho_1 = 0$ and the extreme two terms of inequality (4) are also zero.

§5. We now suppose $f(z)$ is of finite and positive order ρ. Let

$$\alpha = \limsup_{r\to\infty} \left\{ \log M(r)/r^{\rho} \right\}; \ \beta = \liminf_{r\to\infty} \left\{ \log M(r)/r^{\rho} \right\}$$

$$\gamma = \limsup_{r\to\infty} \left\{ \nu(r)/r^{\rho} \right\}; \ \delta = \liminf_{r\to\infty} \left\{ \nu(r)/r^{\rho} \right\}$$

$$L = \limsup_{r\to\infty} \left\{ n(r)/r^{\rho} \right\}; \ l = \liminf_{r\to\infty} \left\{ n(r)/r^{\rho} \right\}$$

We prove some relations between these numbers.

THEOREM 2. *If $f(z)$ be of order ρ, $0 < \rho < \infty$*

(i) $\gamma \geqslant \alpha\rho \geqslant \beta\rho \geqslant \delta$

(ii) $\alpha \geqslant \dfrac{L \log a + l}{a\rho}$ *where*[3] *a is any number* $\geqslant 1$.

---

[1] We suppose that $f(z)$ does not reduce to a constant. Hence $\nu(r) \geq 1$ for all large r.

[2] J. M. Whittaker: The lower order of Integral Functions, *J. L. M. S.*, 8, (1933) 20—27.

[3] See also M. L. Cartwright *Proc. London Math. Soc.* 33. (1932) 209—224 (Theorems 4 and 9).

Proof. Let $a \geqslant 1$ and suppose $0 < \delta < \infty$.

$$\log M\left(ra^{1/\rho}\right) \geqslant A_3 + \int_{r_0}^{ra^{1/\rho}} \frac{\nu(x)}{x} dx$$

where $\nu(x) x^{\rho} > \delta - \varepsilon$ for $x \geqslant r_0$.

Hence $\log M\ (ra^{1/\rho}) \geqslant A_3 + \dfrac{(\delta-\varepsilon)r^{\rho}}{\rho} + \dfrac{\nu(r)}{\rho} \log a.$

$$\therefore \quad \alpha \geqslant \frac{\gamma \log a + \delta}{\rho a}; \quad \beta \geqslant \frac{\delta \log a + \delta}{\rho a}$$

which can be shown to hold also when $\delta = 0$ or ∞.

Taking $a = 1$ we get $\beta \geqslant \delta/\rho$. Similarly we get

$$\alpha \geqslant \frac{L \log a + l}{\rho a}$$

To prove $\alpha \leqslant \gamma/\rho$ we may suppose $\gamma < \infty$.

$$\log M(r) \sim \log \mu(r) = A_4 + \int_{r_1}^{r} \frac{\nu(t)}{t} dt < A_5 + (\gamma + \varepsilon) \int_{r_2}^{r} t^{\rho - 1} dt$$

and hence the theorem is proved.

GLEANINGS

When all things were ready, the moment they (The Athenians) were going to set sail, wholly unsuspected by the enemy, who were far from surmising that they would leave Sicily so soon, the moon was suddenly eclipsed in the middle of the night, and lost all its splendour; which terrified Nicias and the whole army, who, from ignorance and superstition, were astonished at so sudden a change, the causes of which they did not know, and therefore dreaded the consequences of it. They then consulted the soothsayers; and who, being equally unacquainted with the reasons of this phenomenon, only augmented their consternation. It was the custom, after such accidents had happened, to suspend their enterprises but for three days. The soothsayers pronounced that he must not set sail till three times nine days had past; these are Thucydides' words, which doubtless was a mysterious number in the opinion of the people. Nicias, scrupulous to a fault, and full of a mistaken veneration for these blind interpreters of the will of the gods, declared that he would wait a whole revolution of the moon, and not return till the same day of the next month; as if he had not seen the planet very clearly the instant it had emerged from that part which was darkened by the interposition of the Earth's body.

From ROLLIN: *Universal History*

ON THE PROBABILITY OF OBTAINING k SETS OF CONSECUTIVE SUCCESSES IN n TRIALS

BY

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Let p be the probability of the occurrence of an event E in any one trial and q the probability of its failure so that $p+q=1$. The probability of obtaining a run of r or more consecutive successes (the occurrence of the event being counted a success) in n trials is given by * $(1-z_n)$ where z_n is the coefficient of x^n in

$$\frac{1-p^r x^r}{1-x+p^r q x^{r+1}}$$

We shall generalise this result to find the probability of obtaining any number, k say, of sets of r or more consecutive successes in n trials.

Now, consider the ways in which we can get k sets of r or more consecutive successes in $n+1$ trials. This can happen in two mutually exclusive ways.

(i) We may get k sets in $n+1$ trials only.

(ii) We may get k sets in the first n trials and the $(n+1)^{th}$ trial may be a failure if the n^{th} trial is the last of a set of $(r-1)$ consecutive successes, k sets of r consecutive successes having been completed earlier in the series of trials, it being immaterial in any other case whether the $(n+1)^{th}$ trial is a success or failure.

Let $y_{k,n}$ denote the probability of obtaining k sets of r (or more) consecutive successes in n trials. Then the probability of contingency (i) is $y_{k-1,\, n-r}\, qp^r$ because we must have $k-1$ sets in the first $(n-r)$ trials, then a failure and then r consecutive successes. The probability of contingency (ii) is the sum of the following probabilities:

(a) the probability of getting k sets in the first $(n-r)$ trials, then a failure followed by $(r-1)$ consecutive successes and failure at the $(n+1)^{th}$ trial is $y_{k,\, n-r}\cdot q^2 p^{r-1}$;

* J. V. Uspensky, *Introduction to Mathematical Probability* (McGraw Hill, 1937) page 79. See also Whitworth, *Choice and Chance*.

(b) the probability of getting k sets in n trials minus the probability of getting k sets in the first $(n-r)$ trials followed by a failure and $(r-1)$ consecutive successes in order $= y_{n,k} - y_{k,n-r} \cdot qp^{r-1}$.

The sum of the probabilities of contingencies (i) and (ii) must be the probability of getting k sets in $(n+1)$ trials. Thus we have the recurrence relation,

$$y_{k,n+1} = y_{k-1,n-r} \cdot qp^r + y_{k,n-r} \cdot q^2p^{r-1} + y_{k,n} - y_{k,n-r} \cdot qp^{r-1} \quad \ldots \quad (1)$$

This equation can be reduced to

$$\phi_k(x)\,[1 - x - q^2p^{r-1}x^{r+1} + qp^{r-1}x^{r+1}] - y_{k,r}x^r = qp^rx^{r+1}\phi_{k-1}(x)$$

where $\phi_k(x) = y_{k,0} + y_{k,1}x + y_{k,2}x^2 + \cdots\cdots \quad \ldots \quad (2)$

From (2) we obtain

$$(1 - x + qp^rx^{r+1})^k\phi_k(x) = (qp^rx^{r+1})^k\,\phi_0(x)$$

$\phi_0(x)$ is the expression (due to Whitworth) given in the opening para of the paper and

$$\phi_k(x) = \frac{[qp^rx^{r+1}]^k(1 - p^rx^r)}{[1 - x + p^rqx^{r+1}]^{k+1}}$$

The required probability $y_{k,n}$ is thus the coefficient of x^n in $\phi_k(x)$ expanded in ascending powers of x.

GLEANINGS

What is reported of Euclid the Magarian, explains still better how high the passion of Socrates' disciples ran to receive the benefit of his instructions. There was at that time an open war between Athens and Magara, which was carried on with so much animosity, that the Athenians obliged their generals to take an oath to lay waste the territory of Magara twice a year, and prohibited the Magarians to set foot in Attica upon pain of death. This decree could not extinguish nor suspend the zeal of Euclid. "He left his city in the evening in the disguise of a woman, with a veil upon his head, and came to the house of Socrates in the night, where he continued till the approach of day, when he returned in the same manner he came.

From ROLLIN'S *Universal History*

A PROBLEM IN COMBINATIONS

BY

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§1. The problem is "Can m persons keep a watch on m days, n persons keeping a watch daily, so that any two persons go together on two and only two days."

Dr Hansraj Gupta* gave solutions of this problem for $n=3, 4, 5$ and 6. Therein he has given, what are called "elegant solutions" for $n=3, 4$ and 5. For $n=6$ he has not obtained any "elegant solution". Here I show that there can be no elegant solution for $n=6$.

When one day's arrangement is taken, and if the other $(m-1)$ arrangements can be formed by adding $1, 2, \cdots, m-1$, to the members of this arrangement, the solution will be called an "elegant solution". We also know that for the existence of a Solution of the above problem, m must be equal to $\frac{n(n-1)}{2}+1$. We denote m persons by the numbers of $1, 2, 3, \cdots, m$.

§2. $\boldsymbol{n=3, m=4}$. Let the first row of the "elegant solution" be $(1, 2, 2+x_1)$, without loss of generality. The three differences formed out of these three numbers are $1, x_1, x_1+1$. These must neither be equal, nor two of them equal and the third complementary to them. [Two numbers a and b are said to be complementary if $a+b\equiv 0 \pmod{m}$]. Hence for $x_1=1$ or 2 we get the following solution

$$\begin{array}{c} 1, 2, 3, \\ 2, 3, 4, \\ 3, 4, 1, \\ 4, 1, 2. \end{array}$$

§3. $\boldsymbol{n=4, m=7}$. Let the first row be $(1, 2, 2+x_1, 2+x_1+x_2)$. The 6 differences formed out of these 4 numbers are $1, 1+x_1, 1+x_1+x_2, x_1, x_1+x_2$, and x_2; and not more than two of these differences can be either equal or complementary. Here $x_1+x_2\leqslant 5$, since we can arrange the numbers in ascending order of magnitude. The possible sets are (1) $x_1=1, x_2=2$; (2) $x_1=1, x_2=3$; (3) $x_1=2, x_2=3$;

* *The Mathematics Student* Vol. VIII, No. 3, p. 31.

(4) $x_1=3$, $x_2=2$. The sets (3) and (4) give the same solutions as the sets (1) and (2). They are

1, 2, 3, 5,	1, 2, 3, 6,
2, 3, 4, 6,	2, 3, 4, 7,
3, 4, 5, 7,	3, 4, 5, 1,
4, 5, 6, 1,	4, 5, 6, 2,
5, 6, 7, 2,	5, 6, 7, 3,
6, 7, 1, 3,	6, 7, 1, 4,
7, 1, 2, 4,	7, 1, 2, 5,

§4. $n=5$, $m=11$. Let the first row be $(1, 2, 2+x_1, 2+x_1+x_2, 2+x_1+x_2+x_3,)$ where $x_1+x_2+x_3 \leqslant 9$. Imposing the conditions for an elegant solution upon the 10 differences formed out of the above 5 numbers, the following possible sets are obtained:

(1) $x_1=1, \; x_2=2, \; x_3=3$;
(2) $x_1=1, \; x_2=4, \; x_3=3$;
(3) $x_1=2, \; x_2=3, \; x_3=4$;
(4) $x_1=4, \; x_2=3, \; x_3=2$.

The sets (3) and (4) give the same solutions as the sets (1) and (2). They are:

1, 2, 3, 5, 8,	1, 2, 3, 7, 10,
2, 3, 4, 6, 9,	2, 3, 4, 8, 11,
3, 4, 5, 7, 10,	3, 4, 5, 9, 1,
4, 5, 6, 8, 11,	4, 5, 6, 10, 2,
5, 6, 7, 9, 1,	5, 6, 7, 11, 3,
6, 7, 8, 10, 2,	6, 7, 8, 1, 4,
7, 8, 9, 11, 3,	7, 8, 9, 2, 5,
8, 9, 10, 1, 4,	8, 9, 10, 3, 6,
9, 10, 11, 2, 5,	9, 10, 11, 4, 7,
10, 11, 1, 3, 6,	10, 11, 1, 5, 8,
11, 1, 2, 4, 7,	11, 1, 2, 6, 9,

§5. $n=6$, $m=16$. Let the first row, as usual, be $(1, 2, 2+x_1, 2+x_1+x_2, 2+x_1+x_2+x_3, 2+x_1+x_2+x_3+x_4,)$ where $x_1+x_2+x_3+x_4<14$. When the successive 16 differences are formed, it is found after calculation that there is no set of numbers x_1, x_2, x_3, x_4 which gives an elegant solution. Hence there is no elegant solution for $n=6$.

Similarly, by calculation, we may examine the existence of elegant solutions for $n=7$.

SOME ELLIPTIC FUNCTION FORMULAE

BY

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We have $\wp'(u+\omega_1)\cdot\wp'(u+\omega_2)$

$$= \left[-\frac{(e_1-e_2)(e_1-e_3)}{\{\wp(u)-e_1\}^2}\wp'(u)\right]\left[-\frac{(e_2-e_1)(e_2-e_3)}{\{\wp(u)-e_2\}^2}\wp'(u)\right]$$

$$= k\,(e_1-e_2)\{\wp(u)-e_3\}^2,$$

where $k = \dfrac{(e_1-e_2)(e_2-e_3)(e_3-e_1)}{[\{\wp(u)-e_1\}\{\wp(u)-e_2\}\{\wp(u)-e_3\}]^2}\cdot\wp'^2(u)$

$$\therefore\quad \sum\wp'(u+\omega_1)\,\wp'(u+\omega_2) = k\sum e_3^2\,(e_1-e_2)$$

$$= -\frac{\{(e_1-e_2)(e_2-e_3)(e_3-e_1)\}^2\,\wp'^2(u)}{[(\wp(u)-e_1)(\wp(u)-e_2)(\wp(u)-e_3)]^2}$$

$$= -16\,\frac{\{(e_1-e_2)(e_2-e_3)(e_3-e_1)\}^2}{\wp'^2(u)}$$

$$= -\frac{\wp'(u)\cdot\wp'(u+\omega_1)\cdot\wp'(u+\omega_2)\cdot\wp'(u+\omega_3)}{\wp'^2(u)}$$

Thus, we have

$$(1)\quad \frac{1}{\wp'(u)}+\frac{1}{\wp'(u+\omega_1)}+\frac{1}{\wp'(u+\omega_2)}+\frac{1}{\wp'(u+\omega_3)}=0.$$

Also, by differentiation, this gives

$$(2)\quad \frac{\wp''(u)}{\wp'^2(u)}+\frac{\wp''(u+\omega_1)}{\wp'^2(u+\omega_1)}+\frac{\wp''(u+\omega_2)}{\wp'^2(u+\omega_2)}+\frac{\wp''(u+\omega_3)}{\wp'^2(u+\omega_3)}=0.$$

We shall write for shortness,

$$\rho=\wp(u);\ \rho_1=\wp(u+\omega_1);\ \rho_2=\wp(u+\omega_2);\ \rho_3=\wp(u+\omega_3)$$

Then, on differentiating (2) we have

$$\sum\frac{\wp'''}{\wp'^2}-2\sum\frac{\wp''^2}{\wp'^3}=0$$

where, $\sum\dfrac{\wp'''}{\wp'^2}$ means $\dfrac{\wp'''(u)}{\wp'^2(u)}+\dfrac{\wp'''(u+\omega_1)}{\wp'^2(u+\omega_1)}+\cdots+\dfrac{\wp'''(u+\omega_3)}{\wp'^2(u+\omega_3)}$

(i.e.) since $\wp'''=12\,\wp\,\wp'$,

$$\sum\frac{\wp''^2}{\wp'^3}=6\sum\frac{\wp}{\wp'}=0.$$

$$(3)\quad\text{Thus}\qquad \frac{\rho''^2}{\rho'^3}+\frac{\rho_1''^2}{\rho_1'^3}+\frac{\rho_2''^2}{\rho_2'^3}+\frac{\rho_3''^2}{\rho_3'^3}=0$$

On differentiation, (3) gives

$$\sum\left(\frac{2\,\wp''\,\wp'''}{\wp'^{3}}-3\,\frac{\wp''^{3}}{\wp'^{4}}\right)=0$$

(i.e.) $3\sum\dfrac{\wp''^{3}}{\wp'^{4}}=24\sum\dfrac{\wp\wp''}{\wp'^{2}}=24\cdot4=96,$

since, on differentiation of $\sum\dfrac{\wp}{\wp'}=0$, we have $\sum\dfrac{\wp\wp''}{\wp'^{2}}=4.$

(4) Thus, $\quad \dfrac{\rho''^{3}}{\rho'^{4}}+\dfrac{\rho_1''^{3}}{\rho_1'^{4}}+\dfrac{\rho_2''^{3}}{\rho_2'^{4}}+\dfrac{\rho_3''^{3}}{\rho_3'^{4}}=32$

On differentiating (4),

$$\sum\left(3\,\frac{\wp''^{2}\cdot12\,\wp\wp'}{\wp'^{4}}-4\,\frac{\wp''^{4}}{\wp'^{5}}\right)=0$$

(i.e.) $\sum\dfrac{\wp''^{4}}{\wp'^{5}}=9\sum\dfrac{\wp\wp''^{2}}{\wp'^{3}}=9\sum\left\{\dfrac{\wp}{\wp'}\,(4\,\wp(2u)+8\,\wp)\right\}$

$$=36\cdot\wp(2u)\sum\frac{\wp}{\wp'}+72\sum\frac{\wp^{2}}{\wp'}$$

(5) $\therefore\quad \sum\dfrac{\wp''^{4}}{\wp'^{5}}=72\sum\dfrac{\wp^{2}}{\wp'}$

Now, from the result $\sum\dfrac{\wp}{\wp'}=0$, we have $4=\sum\dfrac{\wp\wp''}{\wp'^{2}}$

Differentiating again, $\sum\dfrac{\wp''}{\wp'}=\sum2\,\dfrac{\wp\wp''^{2}}{\wp'^{3}}$

(i.e.) $\qquad \sum\dfrac{\wp\wp''^{2}}{\wp'^{3}}=0$

(i.e.) $0=\sum\dfrac{\wp\wp''^{2}}{\wp'^{3}}=\sum\dfrac{\wp}{\wp'}\,(4\wp\,(2u)+8\wp)$

$$=8\sum\frac{\wp^{2}}{\wp'}+4\wp\,(2u)\sum\frac{\wp}{\wp'}$$

Thus $\qquad \sum\dfrac{\wp^{2}}{\wp'}=0.$

(6) (i.e.) $\dfrac{\rho^{2}}{\rho'}+\dfrac{\rho_1^{2}}{\rho_1'}+\dfrac{\rho_2^{2}}{\rho_2'}+\dfrac{\rho_3^{2}}{\rho_3'}=0$

Thus from (5), we have

(7) $\dfrac{\rho''^{4}}{\rho'^{5}}+\dfrac{\rho_1''^{4}}{\rho_1'^{5}}+\dfrac{\rho_2''^{4}}{\rho_2'^{5}}+\dfrac{\rho_3''^{4}}{\rho_3'^{5}}=0$

Eliminating $\dfrac{1}{\rho'}\ \dfrac{1}{\rho_1'}\ \dfrac{1}{\rho_2'}\ \dfrac{1}{\rho_3'}$ from (1), (2), (3), (7),

we have

$$(8)\qquad \begin{vmatrix} \frac{\rho''}{\rho'} & \left(\frac{\rho''}{\rho'}\right)^2 & \left(\frac{\rho''}{\rho'}\right)^4 & 1 \\ \frac{\rho_1''}{\rho_1'} & \left(\frac{\rho_1''}{\rho_1'}\right)^2 & \left(\frac{\rho_1''}{\rho_1'}\right)^4 & 1 \\ \frac{\rho_2''}{\rho_2'} & \left(\frac{\rho_2''}{\rho_2'}\right)^2 & \left(\frac{\rho_2''}{\rho_2'}\right)^4 & 1 \\ \frac{\rho_3''}{\rho_3'} & \left(\frac{\rho_3''}{\rho_3'}\right)^2 & \left(\frac{\rho_3''}{\rho_3'}\right)^4 & 1 \end{vmatrix} = 0.$$

Also, again, we can show by using the relations (1) (6) and $\sum \frac{\wp}{\wp'} = 0$, the result:

If α, β, γ, δ be ρ, ρ_1, ρ_2, ρ_3 taken in any order then, accents denoting differentiation w.r.t. the argument, we have,

$$\begin{aligned} &-\alpha' \left[\beta\,(\gamma^2-\delta^2)+\gamma\,(\delta^2-\beta^2)+\delta\,(\beta^2-\gamma^2)\right] \\ &=\beta' \left[\alpha\,(\gamma^2-\delta^2)-\alpha^2\,(\gamma-\delta)+(\gamma\,\delta^2-\gamma^2\,\delta)\right] \\ &=\gamma' \left[\alpha\,(\delta^2-\beta^2)-\alpha^2\,(\delta-\beta)+(\delta\beta^2-\delta^2\,\beta)\right] \\ &=\delta' \left[\alpha\,(\beta^2-\gamma^2)-\alpha^2\,(\beta-\gamma)+(\beta\gamma^2-\beta^2\gamma)\right] \end{aligned}$$

which can be rewritten

$$(9)\qquad \begin{aligned} &-\alpha'\,(\beta-\gamma)(\gamma-\delta)(\delta-\beta)=\beta'\,(\alpha-\gamma)(\gamma-\delta)(\delta-\alpha) \\ &=\gamma'\,(\alpha-\beta)(\beta-\delta)(\delta-\alpha)=\delta'\,(\alpha-\beta)(\beta-\gamma)(\gamma-\delta). \end{aligned}$$

If instead of using the equation (5) we use the relation $\sum \frac{\wp''}{\wp} = 0$ and solve as before, we get

$$(10)\qquad \begin{aligned} &-\alpha' \left[\beta\,(\gamma''-\delta'')+\gamma\,(\delta''-\beta'')+\delta\,(\beta''-\gamma'')\right] \\ &=\beta' \left[\alpha\,(\gamma''-\delta'')-\alpha''\,(\gamma-\delta)+(\gamma\delta''-\gamma''\delta)\right] \\ &=\gamma' \left[\alpha\,(\delta''-\beta'')-\alpha''\,(\delta-\beta)+(\delta\beta''-\delta''\beta)\right] \\ &=\delta' \left[\alpha\,(\beta''-\gamma'')-\alpha''\,(\beta-\gamma)+(\beta\gamma''-\beta''\gamma)\right] \end{aligned}$$

From the above results we can also easily obtain the following relations:

$$(11)\qquad \frac{\rho''^5}{\rho'^6}+\frac{\rho_1''^5}{\rho_1'^6}+\frac{\rho_2''^5}{\rho_2'^6}+\frac{\rho_3''^5}{\rho_3'^6}=768\ \wp'(2u).$$

(12)
$$\begin{vmatrix} \rho & \rho^2 & \rho^4 & 1 \\ \rho_1 & \rho_1^2 & \rho_1^4 & 1 \\ \rho_2 & \rho_2^2 & \rho_2^4 & 1 \\ \rho_3 & \rho_3^2 & \rho_3^4 & 1 \end{vmatrix} = 4\rho\,(2u) \begin{vmatrix} \rho & \rho^2 & \rho^3 & 1 \\ \rho_1 & \rho_1^2 & \rho_1^3 & 1 \\ \rho_2 & \rho_2^2 & \rho_2^3 & 1 \\ \rho_3 & \rho_3^2 & \rho_3^3 & 1 \end{vmatrix}$$

(13) If $x' = \wp'\,(u+\omega_1)$; $x'' = \wp''\,(u+\omega_1)$; similarly for y and z; and if we put
$$(\bar{x},\ \underline{y},\ z) = \begin{vmatrix} -x' & x'' & 1 \\ y' & -y'' & 1 \\ z' & z'' & 1 \end{vmatrix},$$

we can prove that:

$$\begin{aligned} &(x\ \ \underline{y}\ \ z) + (\bar{x}\ \ y\ \ z) + (x\ \ y\ \ z) + (x\ \ \bar{y}\ \ z) \\ &\qquad + (x\ \ y\ \ \bar{z}) + (x\ \ y\ \ \bar{z}) \\ &\qquad = 256\,\frac{[(e_1-e_2)(e_2-e_3)(e_3-e_1)]^2}{\wp'^{\,8}\,(u)} \end{aligned}$$

While still at college Gauss had begun those researches in the higher arithmetic which were to make him immortal. His prodigious powers of calculation now came into play. Going directly to the numbers themselves he experimented with them, discovering by induction recondite general theorems whose proofs were to cost even him an effort. In this way he discovered the "gem of arithmetic," "*theorama aureum,*" which Euler also had come upon inductively, which is known as the law of quadratic reciprocity, and which he was to be the first to prove. (Legendre's attempted proof slurs over a crux.)

The whole investigation originated in a simple question which many beginners in arithmetic ask themselves: How many digits are there in the period of a repeating decimal? To get some light on the problem Gauss calculated the decimal representations of all the fractions $1/n$ for $n=1$ to 1000. He did not find the treasure he was seeking, but something infinitely greater—the law of quadratic reciprocity.

E. T. BELL in *Men of Mathematics*.

NOTES AND DISCUSSIONS.

Nurnberg Proof of Cauchy's General Principle of Convergence

In this note, I prove the sufficiency condition directly from a suitable Dedekind's section.

THEOREM: If, given $\varepsilon > 0$, we can find an n_0 such that

$$|\phi(n_2) - \phi(n_1)| < \varepsilon \text{ for all } n_2 > n_1 \geqslant n_0,$$

then the sequence $\phi(n)$ tends to a limit.

Taking $\varepsilon = 1$, we can find an m such that $|\phi(n) - \phi(m)| \leqslant 1$ for all $n \geqslant m$. Hence for all $n \geqslant m$, $\phi(m) - 1 \leqslant \phi(n) \leqslant \phi(m) + 1$.

So the sequence $\phi(n)$ is bounded; in other words, there are two numbers A and B such that $A < \phi(n) < B$ for all n. ... (1)

Divide the real numbers into two class L and R as follows:

If $a \geqslant \phi(n)$ only for a finite number of values of n, put a in the L class.

If $b \leqslant \phi(n)$ only for a finite number of values of n, put b in the R class.

Then if a number r escapes classification, there should be infinite number of values n_3 and n_4 for which $\phi(n_3) < r < \phi(n_4)$. ... (2)

If possible, let two numbers r and s escape classification, where $r > s$.

Then from (2), there are infinitely many values n_1 and n_2 for which $\phi(n_1) < s < r < \phi(n_2)$, and so $\phi(n_2) - \phi(n_1) > r - s$ for infinitely many values of n_1 and n_2. ... (3)

But this (3) is contrary to the condition given in the statement of the theorem.

Hence if at all, only one number will escape classification. Put this number (if it exists) in the L class. Now,

(*a*) Every number in L is less than every number in R;

(*b*) No real number escapes classification.

(*c*) From (1), A belongs to L and B belongs to R; so both the classes exist.

Hence this division is a section. Let α be the number corresponding to this section. Then, if $\varepsilon > 0$, $\alpha + \varepsilon$ belongs to R and $\alpha - \varepsilon$ to L. So, corresponding every $\varepsilon > 0$, there is an n_0, such that $\alpha - \varepsilon > \phi(n) < \alpha + \varepsilon$ for all $n \geqslant n_0$. Hence $\phi(n) \longrightarrow \alpha$ as $n \longrightarrow \infty$. The theorem is thus proved.

Calcutta University. S. SIVASANKARANARAYANA PILLAI.

A Note on Prof. Genese's Theorem and a simple proof

In 1891 Professor Genese of the National College, Aberstwyth-Wales, published the following theorem :—

With any point in the plane of an ellipse as centre two real cricles can be drawn in either of which triangles can be inscribed whose sides touch the ellipse; the radius R of either is given by $(R^2-OS^2)(R^2-OH^2)=4.\ b^2.\ R^2$. (Q. 10879 Vol. LV of *Educational Times*).

In 1912 the same problem was again published by Mr. S. V. Ramamurti (Adviser to H. E. the Governer of Madras) in the following form:—(Q. 16782, Vol. XX. *E. T.* N. S.)

"The condition that an infinite number of triangles can be inscribed in a circle of radius R and centre S, so as to circumcribe a given ellipse of foci F, F′ and minor axis b is

$$(R^2-SF^2)(R^2-SF'^2)=4.\ R^2.\ b^2."$$

But he was told that the theorem had already been propounded and solved. Divested of its conic-sectional garb the Theorem may be restated as follows:—If S and H are two isogonal conjugates w.r.t. a triangle ABC of which O is the cricumcentre $(R^2-OS^2)(R^2-OH^2)=4.\ R^2.\ ll'$ where l and l are the perpendiculars from S and H on any one of the sides of the triangle. Four proofs have been published of this theorem :—(ET. Vol. LV and LVII).

(1) By Professor Greenstreet.

This proof uses trilinear co-ordinates and also trignometry and actually relies on Salmon's *Conic Sections* (Art 131. Ex. 2.)

(2) By Mr. V. Ramaswamy Iyer who uses Analytical Geometry applying the formula for the tangent to a circle from a point and using determinants.

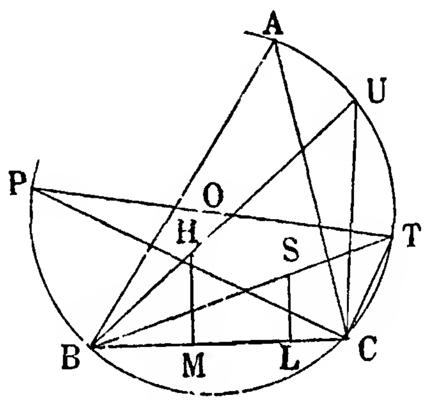
(3) Mr. Heppel

(4) Mr. Gallatly in *The Modern Geometry of the triangle*

These use trigonometrical methods.

The following proof is submitted as a simple proof of the Theorem without requiring more knowledge than that possessed by a School Final student.

Produce BS and BH to meet the circumcircle in T and U. Join TC and UC. Join TO to meet the circumcircle in P. Join PC. Draw SL and HM $\perp$ to BC.



Then, Δs TPC and SBL are similar ($\because \angle$TPC = SBL and TCP = rt. angle = SLB.)

$\therefore$ 2 R. SL = SB. TC.

Similarly, 2 R. HM = HB. UC.

$\therefore$ 4 $R^2 \, ll'$ = SB. HB. TC. UC.

Again, $\angle$TSC = SBC + SCB.

= HBA + HCA. ($\because$ S and H are isogonal)

= BHC − A = BHC − BUC = UCH.

And $\angle$STC = $\angle$CUH.

$\therefore$ Δs TSC and UCH are similar.

$\therefore$ TC. UC = TS. UH

$\therefore$ 4 $R^2 \, l. \, l'$ = BS. BH. ST. HU.

= BS. ST. BH. HU.

$= (R^2 - OS^2)(R^2 - OH^2)$

Madras SIR V. RAMESAM

Note on Weierstrass's Theorem.

MR. E. G. Phillips in his *Course of Analysis* gives the following proof of Weierstrass's Theorem, on page 32, §2-31.

THE THEOREM: Every bounded infinite aggregate has at least one limit point.

PROOF: To fix our ideas, consider only a linear infinite set of points included in the interval (a, b). Take any point c, such that $a < c < b$; then since (a, b) contains an infinite set, there must be an infinite set either in (a, c) or in (c, b) or in both. Suppose for definiteness, that there is an infinite set in (a, c). The set of points x in (a, c) is bounded above by c, so that c, and every number greater than c, is a rough upper bound. The aggregate of rough upper bounds determines a unique number ξ, the upper bound of the set. The number ξ has the properties,

$$\xi \geqslant x$$
$$\xi - \varepsilon < \text{at least one } x;$$

hence in the interval $(\xi - \varepsilon, \xi + \varepsilon)$ there is at least one point of the infinite set in (a, c) other than ξ. The number ξ is therefore a limit point.

Now let us consider the aggregate

$$0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots, \frac{1}{n} \cdots$$

This is an infinite aggregate bounded below by 0, and above by 1. Let c of the theorem be $\frac{1}{2}$, $a = 0$. Then (a, c) contains an infinite aggregate which is bounded by $\frac{1}{2}$ and 0. Therefore, arguing as in the above proof, $\frac{1}{2}$ and every number greater than $\frac{1}{2}$ from the aggregate of rough upper bounds which determines a $\xi = \frac{1}{2}$. But $\xi = \frac{1}{2}$ does not satisfy the condition of an upper bound or limit points. This proof of the theorem, therefore, is not strictly accurate and it fails when c is taken as a point of the aggregate.

Madras VEPA P. SARATHI

A Magic Square of Triangular Numbers.

The following is a magic square formed with the first 64 triangular numbers. The n^{th} triangular number is given by $0+1+2+\cdots+n=\frac{1}{2}n(n+1)$, $n=0, 1, 2, \cdots$. The magic sum, viz, the sum of a row, column or diagonal is equal to 5460 which is itself a triangular number corresponding is $n=104$.

1378	171	496	1653	630	15	36	1081
105	820	703	6	325	2016	1275	210
861	120	3	666	1953	300	231	1326
190	1431	1596	465	10	595	1128	45
0	741	946	91	276	1225	1830	351
1711	406	153	1540	1035	66	21	528
435	1770	1485	136	55	990	561	28
780	1	78	903	1176	253	378	1891

Nurnberg. ALFRED MOESSNER

GLEANINGS

Statisticians, who have gone carefully into figures inform us that if young men who have just received a negative answer to a proposal of marriage (and with these must of course, be grouped those whose engagements have been broken off) 6·08 per cent clench their hands and stare silently before them, 12·02 take the next train to the Rocky Mountains and shoot grizzlies, while 11·07 sit down at their desks and become modern novelists. The first impulse of the remainder—and these, it will be seen constitute a great majority—is to nip off round the corner and get a good stiff drink.

From P. G. WODEHOUSE: *Hot Water.*

To make the idea of this reduction to an absurdity more vivid, the following historical anecdote is told:

A certain eastern king found one of his subjects intoxicated, and worshipping false gods. We are giving the king's reasoning and decision in the matter. Said the king:

1. This man is drunk, therefore he tells the truth;
2. This man is worshipping false gods, therefore he lies;
3. This man is telling the truth and lying at the same time.
4. This man's existence involves an absurdity.
5. Therefore, let this man be executed.

There are many mathematical principles whose demonstration reaches the absurdity point more slowly. If something assumed were to crawl slowly to an absurdity, and reach it after a hundred years, the thing assumed would be proved false. When the egg is hatched out, we discover whether it contains a bird or a snake.

From HAYN: *A Geometry Reader.*

BOOKS RECEIVED FOR REVIEW

1. Scripta Mathematica Library No. 5 *Galois Lectures*, New York, 1941.

2. LEVI, F. W: *Fundamentals of Analysis*, 1942, Calcutta University.

3. ————: *Finite Geometrical Systems*, 1942, Calcutta University.

4. ————: *Algebra*, Vol. I., 1942, Calcutta University.

5. M. K. NARASIMHAMURTI: *A Manual of Geometry*, 1942, Prabhakar Book Depot, Bangalore.

6. CREW, H: *Portfolio of Famous Physicists*, Scripta Mathematica, New York.

7. SURYANARAYANA IYER and SANTIAGO: *Projective Geometry*.

8. RAMESAM, V. and SARATHI, V. P: *A Supplement to Elementary Geometry*, 1942, K. Mahadevan, Madras.

9. SEGRE, B: *The Non Singular Cubic Surfaces*, 1942, Humphrey Milford, Oxford, 15/-

10. SESHADRI, T. N: *Handbook of Physical constants and Mathematical tables*, 1942, S. Viswanathan, Madras.

11. MAHAJANI, G. S: *Lessons in Elementary Analysis*, Third Edition, 1942, Aryabhushan Press, Poona.

12. S. R. GUPTA: *Elementary Analytical Statics*, S. Chand & Co., Delhi, Lahore, 1942.

13. M. V. JAMBUNATHAN: *Elementary Calculus*, Seshadri & Co., Bangalore, 1942, price Re. 1—4—0.

14. ————: *Elementary Analytical Geometry*, Seshadri & Co., Bangalore, 1941, Price Annas Twelve.

Many have imagined that there is one long chain from "The whole is greater than any of its parts," to calculating the ratio between the circumference of a circle and its diameter, and that every principle of the subject is a link in this single long chain. This is false in many ways. Nor will it do to represent the subject by a single tall tree with many branches. A better picture would be a grove of about 30 trees; and even this is incomplete until we admit birds and bees into our grove.

From HAYN: *A Geometry Reader.*

REVIEWS

Two Dimensional Potential Problems connected with Rectilinear Boundaries

BY DR. B. R. SETH, Lucknow University Studies No. XIII, 1939.

A number of problems of mathematical physics: two-dimensional motions of fluids, viscous or not, the torsions of a cylinder, St. Venant's flexure problem, some problems on membranes, etc., depend on the solution of the Problem of Neumann. In two dimensions, this consists in finding a solution of the Laplace's equation which satisfies the boundary condition $\frac{d\phi}{dn} = f$, $\frac{d}{dn}$ being the normal derivative on the boundary and f a function of x, and y depending on polynomials of the second degree at most. Now in applied mathematics it is not enough to know the existence of a solution, some construction of it is needed. In the problems treated by the author, the boundaries are rectilinear and it is possible to find a general method of constructing the solution. This method and its applications to important problems are treated in the book under review.

The method which is proposed is much more general than the method of images of Lord Kelvin, and not so laborious as the method suggested by Sokolnikoff, which is a direct application of the general theory of Potentials where the solution is represented by an integral. It is based on the works of Mitchell, Love, S. C. Kluyver, Leathem, Kotter, E. Trefftz, Bickley and Hodgkinson. Prof. B. R. Seth, to whom valuable contributions on the subject are due, gives a fairly good account of this new method, although I should have preferred him to insist on certain aspects of it a little more.

In essence it consists in finding out the form-finite form of the analytic function whose real part is the required solution. This solution contains a finite number of constants which can be calculated by an elegant conformal representation of the domain on another domain whose boundaries are also rectilinear.

A number of applications are treated in detail. The famous St. Venant's flexure problem is dealt with at great length. Very interesting examples of domains having one or more re-entrant angles are also considered and the student of hydrodynamics will find there many questions about which further research work seems very desirable.

C. RACINE

Handbook of Physical Constants and Mathematical Tables

BY T. N. SESHADRI, Presidency College, Madras. Published by S. VISWANATHAN, 2/10 Post Office Street, Madras G. T. pp. 48—iv.

This handbook is planned on the lines of the well-known Clark's tables. It is intended to furnish data, specially useful for students working in an Indian Laboratory. The long and continued use of Clark's tables would have naturally suggested to any one that an alternative handbook would command very little success. Mr. Seshadri's long experience as a teacher of science has made it possible for him to carry out the apparently impossible task of producing this neatly executed and successful handbook. Its forty eight pages are full of useful information, neatly arranged for quick and easy reference.

Opinions may easily differ regarding the material to be included and the order of presentation. It should however be admitted that the author has done his best to meet the demands of several subjects and different points of view.

A few misprints (*e.g.* on p. 30, $e/m = 5{\cdot}305 \times 10^{-17}$ e.s.u. per gm. *instead of* $5{\cdot}305 \times 10^{17}$ e.s.u. per gm.) should be corrected in the next edition. In the matter of spelling of scientific terms, it would be an advantage to follow some standard publication, like the *Proceedings of the Royal Society.*

The handbook is neatly got up, although one misses in its pages, the elegant printing of Clark's tables. But the publisher has spared no pains to make the handbook neat and attractive, in spite of great handicaps now prevalent in the printing industry.

The author and the publisher are to be congratulated on the high standard of this publication.

S. RAMACHANDRA RAO.

A Supplementary to Elementary Geometry

BY Sir V. RAMESAM, Kt, B.A, B.L., Retired Judge, Madras High Court and V. P. SARATHI, M.A., B.L., Pp. 114. 1942. (Published by K. Mahadevan, Madras) Rs. 2.

This little book furnishes striking evidence of the grip that Geometry can have over one's imagination. Sir Vepa Ramesam, a brilliant student of mathematics, had to submerse all his Mathematics except a little golden streak of Elementary Geometry in his active life as a lawyer and judge, but has now found time to give to the world the benefit of his geometrical assimilations. The reader should be thankful to the authors for the rare and beautiful theorems which are now presented for the first time in a form accessible even to undergraduates. The book also caters to the less ambitious student who is content with theorems that pay in examinations e.g., polarity, coaxal circles, maxima

and minima, properties of triangles, concurrency and collinearity. There is a proper emphasis on the basic theorems in geometry which are given in the early pages of the four books into which the booklet is divided.

The sources of the authors' inspiration are Casey's 'Sequel to Euclid' and the 'Educational Times Reprints', both now inaccessible to the ordinary reader. The Caseyan stunt of introducing Ratio and Proportion, and Circles as early as Books I and II dealing with theorems on straight lines, is adopted with approval and, in the reviewer's opinion, sometimes overdone.

Feuerbach's Theorem and the orthopole are the favourite themes of Sir Vepa who † claims to have collected more than sixty geometrical proofs of Feuerbach's theorem. He is particularly fascinated by Youngman's theorems mentioned in Bk. I, 52 and his own deduction of Feuerbach's theorem (Bk. III, 9) from it. which he claims to be new, (p 56), and asserts as the simplest geometrical proof (*The Math. Student* Vol., IX, p. 130). In Bk. III, 37, it is proved that the nine point circles of AIB, BIC, CIA pass through the in-Feuerbach point, (I being the incentre) much in the same way as Beard's proof as presented in a condensed and modified form in *The Math. Student* Vol. IX pp. 18, 19, where it is utilised for proving Feuerbach's theorem by the addition of one more obvious step. *The authors have not chosen to take this step and furnish* another proof of Feuerbach's theorem which is bound to impress any reader as much simpler, taken all in all, than the deduction of Feuerbach's theorem from Youngman's theorem.

The existence of the 'orthopole' is proved in Bk. I, 32, while further properties are discussed in Bk. III 24, 41, 45, 47, 48 and Bk. IV 21. Cor. 2. With the help of Pascal's theorem, an interesting result is proved, viz., the orthopole of a circumdiameter of A B C with respect to the triangle lies on the "reciprocal" to this diameter. If the equation of a line in areal co-ordinates be $lx+my+nz=o$, the line $x/l+y/m+z/n=o$ is called by Goormaghtigh the "reciprocal" to the first line. These lines are referred to as 'Seitengegengeraden' in the *Encyclopadie der Mathematischen Wissenschaften:* Neuere Driecks Geometrie (see also *J. I. M. S.* Vol. XVIII Part II pp. 106-109). The theorem proved in Bk. IV., 21, Cor. 2 is a particular case of Goormaghtigh's theorem:

The reciprocal line of the image in the circumcentre, of any line m passes thro' orthopole M of m.

Difficult and fundamental theorems on isogonal conjugates due to Genese and Ramaswami Iyer are proved in Bk. III, 49 (the reference to this in the index is an error, the page reference ought to be 85 instead of 86) and 56, involving an abundant use of rectangle properties of circles and similar triangles. There is no reason why they should be placed here, for they may as well occur in Bk. II. quite consistently with the Caseyan scheme which accommodates rectangle properties of circles and similar triangles long before Ratio and proportion and Circles.

† The Mathematical Gazette, Vol. XXV, p. 225.

The book concludes with Pascal's theorem and Desargues' theorem proved by means of Menelaus' theorem, which, however, does not find a place in the booklet. The omission of Ceva's and Menelaus' theorems is unfortunate. Such important theorems deserve attention and elaboration in a book of this kind. The enunciation of Pascal's theorem is indifferent. On the other hand there is a good introduction to Desargues' theorem as a plane section of a space configuration of lines and planes through five points, originally due to Von Staudt. It is to be wondered how many teachers attempt it in the class room.

Regarding elementary theorems the authors have tried to improve on existing proofs, with a good measure of success, witness for example, Bk. II 33. But there is scope for improvement in some other cases.

The proof of the converse theorems in Book II, 14 viz., 'if AC, BD intersect in E such that $AE \cdot EC = BE \cdot ED$, then, A, B, C, D lie on a circle', may be improved thus:

If the perpendicular bisectors of AC, BD meet in S and SE be joined, then $SB^2 - SE^2 = BE \cdot ED = AE \cdot EC = SA^2 - SE^2$, and so $SB = SA = SC = SD$.

This proof is given in the *New Geometry for High Schools* (pp. 517, 518), published in 1928 by Srinivasa Varadachari & Co., Madras.

In Book III, 14 are discussed some loci with precise demarcations. They were first pointed out more than twentyfive years ago in *The Mathematical Gazette* and since then it became a fashion to incorporate them in some text-books. But the precision suddenly stops when it comes to the locus of the 9-points centre, given the base and the vertical angle. Even here, the locus is not a complete circle, being terminated by the mid-points of the joins of the circumcentre to the extremities of the orthocentre-locus.

Again, for the converses of the theorems in Book I 42, 43, 55, direct proofs are available and deserve adoption. According to J. E. Blamey's statements, Herr Lietzmann in *Altes und Neues vom Kreis* (Teubner) has given an excellent proof of the converse proposition of the cyclic quadrilateral and Blamey himself has worked out on Lietzmann's model a proof of the theorem: Given $A\hat{C}B = A\hat{D}B$, and C, D on the same side of AB, then, ABCD is cyclic (Math. Gazette Vol. XXI, p. 231). Here is an illustration. The parallelism in proofs of two dual theorems Book I, 43, 55 is very striking and may be exhibited thus:

If $\hat{A}+\hat{C}=\hat{B}+\hat{D}$ in a quadrilateral ABCD, then a circle can be circumscribed about it.	If $AB+CD=AD+BC$ in a quadrilateral, then a circle can be inscribed in it.
Proof:—Let $A>B$; then $A-B = D-C$, and $D>C$. At B make $A\hat{B}O = D\hat{A}B$, and at C make $D\hat{C}O = A\hat{D}C$.	Proof:—Let $AB>BC$; then $AB-BC = AD-CD$, and $AD>CD$. On BC produced take E so that $BE=BA$; on DC produced take F so that $DF=DA$.

Then OB = OC, and the perpendicular bisectors of AB, BC, CD bisect also the angles of the triangle formed by OB, OC and AD, and meet at a point equidistant from A, B, C, and D.

Then CE = CF and the bisectors of the angles ABC, BCD, CDA are the perpendicular bisectors of the sides of AEF and so meet at a point equidistant from AB, BC, CD, and DA.

On the whole, one can have nothing but admiration for the authors to whom this work must have been a labour of love, and who have demonstrated the possible heights one can reach with such limited equipment as rectangle properties of circles and similar triangles. Some of the rarest gems in the literature of the subject, it has been found possible to include in this work of less than six score pages expounding nearly two hundred theorems great and small. A careful study of the work is sure to inspire the reader to do a bit of geometrical mountaineering himself and attempt to visit the immortal peaks where Pascal, Steiner, and Hamilton hold their sway.

Mysore. A. A. KRISHNASWAMI AYYANGAR

Portraits of Famous Physicists with biographical accounts,

BY HENRY CREW. Scripta Mathematica, New York, 1942, Price $ 3.75.

The Mathematical world already owes a debt of gratitude to the organizers of the *Scripta Mathematica* for a series of "Portraits of eminent mathematicians," each accompanied by a brief biographical sketch. The present portfolio of a dozen of the world's great physicists is issued as the fourth of the series on "pictorial mathematics" and maintains the same high standards that one has learnt to associate with the earlier series. The pictures are, as usual, on art paper while the biographical details are on a separate folder illustrated, in many cases, with sketches relating to the life of the scientist or a facsimile reproduction of his writings.

The series begins with three versatile men who achieved greatness in several fields, notably in Astronomy and Physics—Galileo, Christiaan Huygens and Sir Isaac Newton. Galileo with a telescope made by himself discovered spots on the sun, 4 of Jupiter's satellites, and noted the "triple nature of Saturn" which remained a puzzle for 50 years till Huygens resolved the appendages into the rings of Saturn. Galileo died on 8—1—1642, three days after the birth of Newton and yet he had seized on momentum as the fundamental quantity in mechanics and enunciated the first two laws of motion. Huygens will always be associated with the wave front and optical phenomena like double refraction, but his great contribution to Science was the pendulum clock invented at the age of 28. The portrait of Newton is the one by Vanderbank in the National Museum gallery, London and shows him in a thoughtful mood seated in front of a open book and a star globe.

England, which heads the list of portraits, makes two more contributions in the persons of Michael Faraday who unaided by academic training discovered the fundamental principles upon which the dynamo and the telephone depend and upon which Maxwell erected the electromagnetic theory of light and guided Hertz to the discovery of wireless telegraphy, and J. P. Joule who just a hundred years ago (1843) measured the mechanical equivalent of heat. The frugal Scotch make but one contribution, but he is the immortal James Clerk Maxwell after whom the unit of magnetic flux was named in 1932 as a mark of international appreciation. The Germans are represented by H. R. Hertz and by R. J E Clausius the inventor of the concept of entropy in thermodynamics. We meet two French Scientists-Andre Marie Ampere whose rule regarding the magnetic field due to a current element must be well known to students of Physics, and Augustin Fresnel who died in 1827 at the age of 39 after having established the wave theory of light on an experimental basis.

Of those who hail from the land of the *Scripta* we have but two, Josiah Willard Gibbs who made valuable contributions to thermodynamics but comes close to the student world through his classical work on "Vector Analysis" revised by one of his ablest students E. B. Wilson, and Henry Augustus Rowland whose grating has made spectroscopy what it is to-day.

These portraits were planned in 1937 and took account only of outstanding physicists not then living which explains the absence of Sir J. J. Thomson and Lord Rutherford. I have not been able to discover the order of arrangement of the portraits, but if arranged in the order of the dates of birth, there is only one inversion at the very end.

A. N. RAO.

Lessons in Elementary Analysis

Third Edition by G. S. MAHAJANI, Ph D. (CANTAB), Principal, Fergusson College, Poona. 1942, pp. 294 Price Rs. 6-4-0.

It is a regrettable fact that there is no suitable text-book by Indian authors covering the syllabus in Analysis for the Honours degree courses of Indian Universities. The book under review rectifies this defect to some extent. The author points out in his preface to the first edition that the book was written mainly for the pass course students. But the book as it stands is more suited to the Honours than to pass students even though it does not cover completely the whole course for Honours students. The presentation of the topics covered by the book is both lucid and precise.

A few details deserve mention. In the definition of the limit of a function $f(x)$ at a point $x=a$ (p. 35), the author does not explicitly omit the value of $f(x)$ at $x=a$. He points out that this is to be the case in his later remarks. But a reader might be led to think after a perusal of the whole matter,

that the point raised above is not an essential one, and the author points out the fact somewhat casually. On the other hand, no amount of emphasis will be too much in drawing the attention of students to the fact that the notion of a *limit at a point* is *entirely independent* of the *value of the function at that point.* This being so, it is desirable that the above point be explicitly mentioned in the definition of the limit. Secondly, the chapter on limits would be more complete if the author had added a few elementary notions regarding point sets, limit points and their relation to upper and lower limits and a statement and proof of the conditions for the existence of the limit of a function at a point.

The notion of an infinitesimal as something which vanishes ultimately appears to be superfluous once the idea of a limit has been made precise. The orders of infinitesimals are merely measures of the rapidity of approach to a limit and belong properly to the domain of ideas like the orders of infinity. All the results in chapter III could be defined or proved with the notion of a limit alone. The so called indeterminate forms also come under the notion of limits. Lastly, the chapter on Riemann Integration would gain by a statement and proof of Darboux's theorem.

As already pointed out, the book, is well-written; the printing is very neat; and it satisfies, though partially, a really felt need of the Honours students of Indian Universities for a good text-book on Analysis.

V. G.

GLEANINGS

Probably the most striking case in history is that of the Bernoulli family, which in three generations produced eight mathematicians, several of them outstanding, who in turn produced a swarm of descendants about half of whom were gifted above the average and nearly all of whom, down to the present day, have been superior human beings. No fewer than 120 of the descendants of the mathematical Bernoullis have been traced genealogically, and of this considerable posterity the majority achieved distinction—sometimes amounting to eminence in the Law, Scholarship, Science, Literature, the Learned Professions, administration, and the arts. None were failures. The most significant thing about a majority of the mathematical members of this family in the second and third generations is that they did not deliberately choose mathematics as a profession but drifted into it in spite of themselves as a dipsomaniac returns to alcohol.

From BELL, *Men of Mathematics.*

QUESTIONS FOR SOLUTION

1808. (B. SEETHARAMA SASTRY): Find a function $\phi(x,y)$ such that the plane transformation T defined by

$$\xi = x^p y^q, \ \eta = \phi(x,y) \ (p,q>0)$$

may be a cyclic transformation of the third order.

1809. (SAHIB RAM MANDAN): Show that the vertices of the 8 cones belonging to the system of quadrics $\lambda^2 S_1 + \lambda S_2 + S_3 = 0$ where S_1, S_2, S_3 are the point-equations of any 3 quadrics, form an associated set of 8 points.

1810. (SAHIB RAM MANDAN): A variable conic of a confocal system meets a given line in P,Q. Prove that the normals at P,Q to the conic meet on a fixed line. Identify this line.

1811. (A. O. AUGUSTINE): Let S_1, S_2, S_3 the three given conicoids, and A_{ij}, B_{ij}, C_{ij} and D_{ij} the vertices of the common self-conjugate tetrahedron of S_i and S_j. Prove that a unique conicoid exists which passes through the 9 points

$$A_{23}, B_{23}, C_{23}, D_{23}, A_{13}, B_{13}, C_{13}, A_{12}, B_{12}.$$

1812. (NATHAN ALTSHILLER-COURT): University of Oklahoma, U. S. A.

The lines AO, BO, CO, DO joining the vertices of the tetrahedron (T) = ABCD to the point O (not in a face of the tetrahedron) meet the respective faces of (T) in the points K, L, M, N, and the planes LMN, MNK, NKL, KLM in the points P, Q, R, S. Let Q_a, R_a, S_a be the traces of the lines AQ, AR, AS in the plane BCD; R_b, S_b, P_b the traces of the lines BR, BS, BP in the plane CDA, etc. (i) The four triads of planes BCD, LMN, $P_bP_cP_d$; CDA, MNK, $Q_cQ_dQ_a$; DAB, NKL, $R_dR_aR_b$; ABC, KLM, $S_aS_bS_c$ are coaxal; (ii) The four axes of these four triads are coplanar.

[*Note.* A similar problem in the plane was discussed in the Educational times, Reprints, vol. LXX (1899), p. 112, Q. 13930.]

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THE CALENDAR *

BY

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1. Our Calendar is the result of ages of human thought and experience. The ancient Egyptians, who built the pyramids about 5000 years ago, were busy solving the Calendar Problem, and it cannot be said to have been completely solved even now. I shall confine myself to the three Calendars in use at present in Northern India, viz. the Christian, the Vikrama or the Hindu, and the Hijri Calendars and shall try to explain the main principles on which they are based.

2. It is a matter of common knowledge now that the revolution of the earth round the sun is responsible for the changes in seasons. The ancients did not know this. When they took to agriculture for their living, they wanted to know when they were to sow and when the rains were to be expected. The moon with her changing phases came to their help. They found out that between one rainy season and the next there was a certain number of new moons. It was soon noticed that this information was not sufficient. Having no observatories or scientific instruments, it took them very long to discover that the seasons came in a cycle of about 360 days, or that the sun rose at the same place after an interval of about 360 days. Thus the track of the sun in the heavens was mapped out in 360 steps each corresponding to a day and a night. Incidentally it may be remarked that our present method of dividing the circumference of a circle into 360 equal "degrees," is a natural development of this simple procedure. Later it was found that this estimate was not correct. The addition of five feast days to the twelve Egyptian months of 30 days each seems to point to the recognition of this error. This new estimate is nearly correct. The Hindus estimated the length of the year at 365 days 6 hours and 12 minutes nearly. Considering the difficulties under which they laboured and an almost entire lack of scientific instruments available to modern investigators, this was a great achievement indeeed. We know now that this is slightly in excess, as it takes the earth 365 days 5 hours 48 minutes and 46 seconds, to go round the sun and come back to its original position in its orbit. This period of time is called the "Solar Year." Again it was found that between one new moon and the next one the interval was about $29\frac{1}{2}$ days. This period came to be called a "month" on account of its associ-

* Based on a paper read at a meeting of the Study Circle, Hoshiarpur.

ation with the moon. In fact a lunar month is 29 days 12 hours 44 minutes and 3 seconds approximately. Gradually, therefore, the word "month" came to be used for a period of about 30 days.

3. It is well-known that the length of the day the period of time between one sun-rise and the next is not always the same except at places on the equator. We divide a day into 24 hours, each hour into 60 minutes and each minute into 60 seconds. The Hindus have a better and more uniform way of dividing the day into 60 gharis, each ghari into 60 pals and each pal into 60 vipals. A ghari is 2/5 of an hour, a pal 2/5 of a minute and a vipal 2/5 of a second. It will be interesting to note that the ratio 2/5 seems to be of some significance in dealing with Indian measures, thus one Kachcha seer is 2/5 of 1 Pakka seer. Thus understood a ghari may be called a Kachcha hour, a pal a Kachcha minute and a vipal a Kachcha second.

4. The Christian Calendar.

The present Christian year is "solar." It was once "Luni-Solar," i.e. it followed the moon so far as the months were concerned and the sun for the seasons. At intervals an extra month was intercalated in order to keep the calendar in pace with the seasons. The present Hindu "Luni-Solar" Calendar does the same. Julius Ceasar, with the advice and assistance of Sosigenes, fixed the mean length of the year at 365¼ days and laid down that every fourth year should have 366 days others having 365. This made the mean year too long by 11 minutes 14 seconds, the error accummulating to nearly 3 days in the course of 400 years. Pope Gregory XIII therefore directed in March, 1582 that 10 days be suppressed in the Calendar and ordered that February should have only 28 days in all centenary years excepting those that are multiples of 400. Thus the year 1900 was not a leap year, while the year 2000 will be. Even this arrangement leaves the average year too long by 26 seconds. In 3323 years the excess will amount to one day. It has, consequently, been proposed to correct the "Gregorian Rule" by making the year 4000 an ordinary instead of a leap year. It would have been very much better if Pope Gregory XIII had ordered that February should have 29 days in all years which are multiples of 4 but not of 128. We would thus have had 31 leap years, in 128 years. This would have left the average year in defect only by 1 second. It is a pity that he did not so ordain.

The Christian Calendar has 12 months of unequal lengths; seven having 31 and four 30 days each, and February having 28 in the ordinary and 29 in the leap year. It is believed that the months were once alternately of 31 and 30 days each except February which had 29 in the ordinary and 30 days in the leap year. It may be of interest to mention that the year originally began with March; September, October, November and December were so named because they were respectively the 7th, 8th, 9th and the 10th months of the year. July was called Quintilis and August Sextilis. Quintills was re-named July in order to please Julius Caesar while Sextilis was rechristened August to gratify

Augustus. It is further believed that August was allotted 31 days simply because Julius Caesar's July had 31. Augustus could not bear that his month should have fewer days than Julius's. Several attempts were made to change the names of the other months to humour kings and emperors, but evidently they failed. When August was given 31 days, poor February (the last month of the year) had another day knocked off. In recent years attempts have been made to revise the Calendar to suit the needs of the commercial and wage-earning communities. It has, for instance, been proposed to have 13 months of 28 days each, with a general holiday on the New Year's Day which is to have no date or dayname. The Leap Year's Day is to be a holiday in the same manner. Every month is to begin with a Sunday. National Festivals are to be inserted on Saturdays or Mondays once for all, so that the people may combine the holidays to be better able to enjoy themselves. One reason why these reforms have not been accepted is that the superstitious West does not like the idea of having 13 months on account of the ominous associations of that number.

To sum up, the main points about the present Christian Year are:—

(1) It is "Solar". (2) The lengths of the months are purely arbitrary and have no scientific basis. (3) The quarters are not of the same length. (4) The day is from mid-night to mid-night.

It may be remarked that the Gregorian Reform was not immediately accepted. Russia was the last to do so under the Soviet Government. It appears that "inertia" is as much applicable to the laws of the mind as to those of the physical world.

5. The Vikrama Calendar.

It is of two different types:— (1) Solar (2) Luni-Solar. In the Luni-Solar Calendar we have "tithis" (dates of the moon), as the basis of reckoning. Every three years a month is intercalated in the lunar year so as to make it keep pace with the solar year.

The Solar year has, as already stated, 365 days, six hours, 12 minutes and 36·56 seconds, according to the present Surya Siddhanta. It is divided into 12 months of different durations. The time taken by the earth in revolving round the sun through an angle of 30° is termed a "month" in the solar year of the Hindus. If the path of the earth were a circle with the sun at the centre, the months would all be of equal length; but as the orbit of the earth is an ellipse and the sun is in one of the foci, the earth takes different periods of time in revolving through angles of 30° round the sun at different times of the year. The months have different lengths on this account. Thus according to the present Surya Siddhanta the length of the month of *Asaḍh* is 31 days, 15 hours 28 minutes and 24 seconds approximately, whereas the length of the month of *Phalguṇa* is only 29 days, 19 hours, 41 minutes 12 seconds. The exact time at which a new month begins is termed as "Samkranti". The day is

reckoned from sun-rise to sun-rise, the point at the equator on the longitude of the place being taken as the basis for the reckoning. The ancient Indian almanacs took a place on the equator at the longitude of Ujjain as their basis and all times were calculated from the sun-rise at the place. Ujjain may be called the Greenwich of India. As a convention it was enjoined that the first of the month should be considered to have fallen on the day during the course of which the "Samkranti" occurs. Thus if the Samkranti were to occur even a few minutes before equatorial sunrise tomorrow, today will be regarded as the first of the month. This convention is very sound and leaves no scope for leap years. It will thus be clear that Hindu months can have days from 29 to 32. In general the summer months have more days than the winter months.

The Hindu Calendar needs reform in two directions. First, it should begin with the "vernal equinox" as it was actually intended to. Secondly, the length of the year should be corrected. The length of the year as stated in the Surya Siddhanta is in excess of the correct length by nearly 24 minutes. We are thus moving ahead at the rate of one day in every 60 years. If the necessary reform is not affected the first of *Vaishakh* may come in winter one day. It is time that we rectified the error. Opposition there is bound to be, but the earlier an error is corrected the better. In all other respects the Hindu Calendar is more scientific and more in conformity with nature than the Christian Calendar in use at present; because the months are not arbitrary in length, and because the day begins with the most charming natural event—the sun-rise, and not at mid-night when the world is plunged in darkness.

6. The Hijri Calendar.

The era commences with the flight of the prophet from Mecca to Medina, which took place on Friday, the 19th July, 622 A. D. according to the reformed calendar. The Hijri year is purely lunar. It has 12 months of 29 days, 12 hours, 44 minutes and 3 seconds each. In practice the months have 30 and 29 days alternately. This leaves the year too short by 8 hours, 48 minutes and 36 seconds. In 30 years the difference amounts to about 11 days. Umar Khayyam, the poet and mathematician, therefore hit upon a cycle of 30 years in which 19 years have 354 days each and 11 years have 355 days each. The intercalary years are the 2nd, 5th, 7th, 10th, 13th, 16th, 18th, 21st, 24th, 26th, and 29th. This arrangement is very satisfactory in so far as it leaves the average Hijri year too short only by 36 seconds. However, Umar Khayyam's plan seems never to have been adopted in practice and the appearance of the moon has always been a natural event of importance in the Hijri year. The month begins with the appearance of the moon and the first of the month falls on the day following the appearance of the moon. In the Hijri Calendar the day is calculated from sunset to sunset. It will, thus, be readily seen that there is an element of great uncertainty in the Hijri Calendar and even the almanacs for any year are not wholly trustworthy. I have made an attempt*

* See my paper on the "Indian Calendar" in *Popular Astronomy*, 1934, Vol. 42 pp. 82-84.

towards removing this uncertainty by taking into consideration the 36 seconds in a year left out by Umar Khayyam. It is satisfying to note that my calendar for Hijri years gives results which are more in agreement with the observations and the prevailing conventions than those given by the almanac makers. It will not be out of place to remark that the Hijri months are not arbitrary; they are wholly based on natural events. Since the Hijri year is purely lunar, its months give no idea of the seasons. In about 33 years' time the Hijri months make a round through the solar year.

GLEANINGS

HEIGHTS AND DISTANCES

He further exhibited with great pride a mass of figures which he had worked out upon the mirror of Mr. Purvale's chiffonier with a stick of shaving soap. "So that," he went on, "by further measuring the exact deadline between where the bullets entered the room, to where they struck the wall, and then right-angling the tangent we have got the exact point of line of fire as so many feet higher, which'd make it some window on the other side of the street, and so many feet to the right or left of that, as it may be. You can see that clearly, can't you?" he demanded. "I tell you, my lad, you learn something when you've got to read for the Army. Even whether you pass your exams or not," he amended somewhat lamely.

From: J. G. BRANDON, IN *The Glass Dagger*.
Per A. A. Krishnaswami Ayyangar, *Mysore*.

The property of attraction is inherent in the earth. By this property, the Earth attracts any unsupported heavy thing towards it. That thing appears to be falling, but it is in a state of being drawn to the Earth. The etherial expanse being equally outspread all round, where can the Earth fall? It is manifest from this that neither can the Earth by any means fall downwards nor the men situated at the distance of a fourth part of a circumference from us or in the opposite hemisphere.

From BHASKARA: *Goladhyaya.*

As the one-hundred part of the circumference of a circle is scarcely different from a plane and the Earth is an excessively large body and a man exceedingly small in comparison, consequently the whole visible portion of the Earth appears to man on its surface to be perfectly plane.

From BHASKARA: *Goladhyaya.*

SOME THEOREMS CONCERNING QUINTICS INSOLUBLE BY RADICALS

BY

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In what follows all the latin letters denote integers.

The equation

$$x^5+cx+d=0 \qquad \dots \quad (1)$$

if irreducible, is insoluble by radicals if

$$(z-c)^4(z^2-6cz+25c^2)=5^5d^4z \qquad \dots \quad (2)$$

has an integral root. (Cajori's, *Theory of Equations*). Recently we* have constructed similar tests, for the equations

$$x^5+ax^2+bx+c=0 \qquad \dots \quad (3)$$

and

$$x^5+ax^3+bx^2+cx+d=0 \qquad \dots \quad (4)$$

The following theorems concerning quintics insoluble by radicals are easy to prove, but so far as we know, they have never been stated before.

THEOREM 1. $x^5+ax^3+bx^2+cx+d=0$ (if irreducible) is insoluble by radicals if

$$a \equiv b \equiv 0 \pmod 2$$

and

$$c \equiv d \equiv 1 \pmod 2.$$

This is easily deduced from the condition analogous to (2) for (4). As a special case it follows that $x^5+cx+d=0$ is insoluble by radicals is $c \equiv d \equiv 1 \pmod 2$.

THEOREM 2. $x^5+cx+d=0$ is insoluble by radicals if c is odd and not divisible by any prime$\equiv 3 \pmod 4$.

Proof: From equation (2) we have that $z \mid 25c^6$.

Hence z must be odd and $\equiv 1 \pmod 4$; so that the left side of (2) is $\equiv 0$ (mod. 16) and d must be even.

* S. Chowla, in an unpublished note, has obtained the condition analogous to (2) for (3), and Y. Bhalotra for (4) in *The Mathematics Student*, Dec. 41.

It follows that

$$z^2 - 6cz + 25c^2 \equiv 0 \pmod{16}$$

i.e. $$(z - 3c)^2 + 16c^2 \equiv 0 \pmod{16}.$$

Hence, $$z \equiv 3c \pmod 4$$
$$\equiv 3 \pmod 4,$$

contrary to $z \equiv 1 \pmod 4$, proved earlier.

THEOREM 3. $x^5 + cx + d = 0$ is insoluble by radicals if (*i*) c is a prime $\not\equiv 1 \pmod 5$ and (*ii*) $(d, c) = 1$.

Proof. Let us write $c^{\alpha} \| z$ to denote $c^{\alpha} | z$ but $c^{\alpha+1}$ is not a divisible by z.

Then it is clear from the equation (2) that if $\alpha > 0$, then $\alpha \geqq 6$.

Now $z | 25c^6$.

$\therefore$ z can have only the values
$z = 1,\ 5,\ 5^2,\ c^6 \cdot 5,\ c^6 \cdot 5^2.$

Let us suppose that $z = 5c^6$. Then (2) becomes

$$(5c^5 - 1)^4\ (5c^{10} - 6 \cdot c + 5) = 5^5\ d^4$$

Now $5c^5 - 1 \not\equiv 0 \pmod 5$

$$5c^{10} - 6 \cdot c^5 + 5 \not\equiv 0 \pmod 5 \text{ if } c \neq 5.$$

If $c = 5$, then

$$5c^{10} - 6c^5 + 5 = 5\ (5^{10} - 6 \cdot 5^4 + 1)$$

and $$5^{10} - 6 \cdot 5^4 + 1 \not\equiv 0 \pmod 5$$

In either case $5c^{10} - 6c^5 + 5 \not\equiv 0 \pmod{5^5}$

Hence the left side $\not\equiv 0 \pmod{5^5}$ while
right side $\equiv 0 \pmod{5^5}$.

Hence $z = 5c^6$ is not permissible.

If $z = 5^2 \cdot c^6$, (2) becomes

$$(5^2\ c^5 - 1)^4\ (5^2 \cdot c^{10} - 6 \cdot c^5 + 1) = 5^8\ d^4.$$

Now $5^8 c^5 - 1 \not\equiv 0 \pmod 5$

and
$$
\begin{aligned}
5^8 c^{10} - 6c^5 + 1 &\equiv 1 - 6c^5 \pmod 5 \\
&\equiv 1 - 6c \pmod 5 \\
&\not\equiv 0 \pmod 5.
\end{aligned}
$$

Hence the left side cannot contain the factor 5 which exists on the right side.

Similarly we can show that other values of z do not hold.

Hence the theorem enunciated.

GLEANINGS

If a piece of cloth be cut in a circular form with a diameter equal to half the circumference of the sphere, then half the sphere will be entirely covered by that circular cloth and there will be some cloth to spare. As the area of this piece of cloth is to be found nearly $2\frac{1}{2}$ times the area of a great circle of the sphere, and the area of the piece of cloth covering the other half of the sphere is also the same, the area of the whole sphere cannot be more than 5 times the area of the great circle of the sphere. How then has he (Lalla) multiplied the area of the great circle of the sphere by the circumference to get the superficial contents of the sphere? As the area of the great circle of the sphere multiplied by the circumference is without reason, the rule given by Lalla is wrong and consequently surface area of the Earth is wrong.

From BHASKARA : *Goladhyaya.*

In answering this question Professor C. E. M. Joad says that because of science we have been brought to think that a thing to be real must be comprehended by our sense organs like touch, sight, etc. This limitation of science is unwarranted. There are other orders of Reality, which though not material, are none the less real. To these belong mathematics, music and chess. Continues the Professor, "It may be that the soul has inhabited such an order of Reality before it was incarnated in the body and brings with it to this world a memory of the harmonies of sound and pattern and combinations of number which exist in those orders of Reality: the chess genius has perhaps never perfected himself in the pursuits of or obtained familiarity with the things of the terrestrial sensory orders of Reality: hence he is no good at business or the professions.

T. A. KRISHNAMACHARIAR IN *The Hindu.*

ON SYMMETRIC POLYNOMIAL FUNCTIONS OF ZEROS OF POLYNOMIALS

BY

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The object of this note is to give a proof of the well known fundamental theorem on polynomial symmetric functions of zeros of polynomials. The statement of the theorem as it is given in this note is slightly sharper than what one meets with in many elementary textbooks and the proof avoids any reference to the derivative of a polynomial. It is possible that the undergraduate students who first calculate, in terms of the coefficients of a polynomial $P(z)$, the value of a number of simpler symmetric functions of the zeros of $P(z)$ will find the proof given here to be more 'natural'.

Let

$$P(z)=z^n+a_1 z^{n-1}+\cdots+a_n$$

and

$$R=\sum \alpha_{r_1}^{\varepsilon_1}\alpha_{r_2}^{\varepsilon_2}\cdots\alpha_{r_t}^{\varepsilon_t}$$

where $r_1, r_2, \cdots r_t$ are distinct (positive) integers, $\alpha_1, \alpha_2, \cdots \alpha_n$ are the n zeros of $P(z)$, $\varepsilon_1, \varepsilon_2, \cdots \varepsilon_t$ are non-negative integers, $1 \leqslant t \leqslant n$, $1 \leqslant r_k \leqslant n$ $(k=1, 2, \cdots t)$, and the summation is over all such permutations of the suffixes as give rise to different terms. For example, if $n=3$ then we can take R to be such functions as

$$\alpha_1\alpha_2\alpha_3,$$

$$\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3,$$

$$\alpha_1^2\alpha_2+\alpha_1^2\alpha_3+\alpha_2^2\alpha_1+\alpha_2^2\alpha_3+\alpha_3^2\alpha_1+\alpha_3^2\alpha_2.$$

and so on. Such symmetric functions of $\alpha_1, \alpha_2, \alpha_n$ are called here typical symmetric functions; plainly, any symmetric polynomial in $\alpha_1, \alpha_2, \cdots \alpha_n$ is a linear function of typical symmetric functions. We represent the typical symmetric function R by $(\varepsilon_1, \varepsilon_2, \cdots \varepsilon_t)$, and suppose, without loss of generality, that $\varepsilon_1 \geqslant \varepsilon_2 \geqslant \cdots \geqslant \varepsilon_t$. The degree of R is defined to be $\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_t$. We specify below a scheme for enumerating (i.e., ranking) typical symmetric functions. If R_1 and R_2 are two typical symmetric functions of degrees d_1 and d_2

respectively and $d_1 \neq d_2$ then R_1 is of a higher or lower rank than R_2 according as $d_1 > d_2$ or $d_1 < d_2$; if $d_1 = d_2$ but R_1 and R_2 are distinct,

$$R_1 = \{\varepsilon_1, \varepsilon_2, \cdots \varepsilon_t\}, \; R_2 = \{\eta_1, \eta_2, \cdots \eta_s\},$$

$(\varepsilon_1 \geqslant \varepsilon_2 \geqslant \cdots \geqslant \varepsilon_t, \; \eta_1 \geqslant \eta_2 \geqslant \cdots \geqslant \eta_s)$ then R_1 is of a higher or lower rank than R_2 according as the first nonvanishing number of the set $\varepsilon_1 - \eta_1, \varepsilon_2 - \eta_2, \cdots$ is positive or negative. For example, if $n = 3$ then the enumeration of typical symmetric functions is as follows :—

$$T_1 = \sum \alpha_r = \alpha_1 + \alpha_2 + \alpha_3,$$
$$T_2 = \sum \alpha_{r1}\alpha_{r2} = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3,$$
$$T_3 = \sum \alpha_r^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$$
$$T_4 = \sum \alpha_{r1}\alpha_{r2}\alpha_{r3} = \alpha_1\alpha_2\alpha_3$$
$$T_5 = \sum \alpha_{r1}^2\alpha_{r2} = \alpha_1^2\alpha_2 + \alpha_1^2\alpha_3 + \alpha_2^2\alpha_1 + \alpha_2^2\alpha_3 + \alpha_3^2\alpha_1 + \alpha_3^2\alpha_2.$$
$$T_6 = \sum \alpha_r^3 = \alpha_1^3 + \alpha_2^3 + \alpha_3^3.$$
$$T_7 = \sum \alpha_{r1}^2 \alpha_{r2} \alpha_{r3} = \alpha_1^2 \alpha_2 \alpha_3 + \alpha_2^2 \alpha_1 \alpha_3 + \alpha_3^2 \alpha_1 \alpha_2, \text{ and so on.}$$

THEOREM :—*The value of any typical symmetric function R is equal to $Q_R(a_1, a_2, \cdots a_n)$ where $Q_R(a_1, a_2, \cdots a_n)$ is a polynomial in $a_1, a_2, \cdots a_n$ and has integer coefficients.*

Remark. It is obvious that the same conclusion holds for any symmetric polynomial function provided that its coefficients are integers.

Plainly the theorem is true for T_1, and T_2 and, more generally, for such T's as are of the form $\{1, 1, \cdots\}$. We prove the theorem by induction. Suppose now that the theorem is true for $T_1, T_2, \cdots T_m$, and let

$$T_{m+1} = \{\varepsilon_1, \varepsilon_2, \cdots \varepsilon_t\} \qquad (\varepsilon_r \geqq 1, \; r = 1, 2, \cdots t).$$

It is easy to verify that

$$T_{m+1} = \{\varepsilon_1 - 1, \varepsilon_2 - 1, \cdots \varepsilon_t - 1\} \times \{1, 1, \cdots 1\} - \sum_{r=1}^{m} C_r T_r$$

Where the number of 1's in the second factor of the product is equal to t, and $c_1, c_2, \cdots c_m$ are (non-negative) integers which depend on n and m. From the inductive hypothesis we see that the right hand side of the above equation is a polynomial in $a_1, a_2, \cdots a_n$ and that its coefficients are integers. This completes the proof of the theorem.

A PORISM ON ELEVEN SPHERES

BY

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1. Let (R) be the orthogonal sphere of the four spheres (A), (B), (C), (D); centers A, B, C, D, and (L) another given sphere. The sphere (A′) is constructed orthogonal to (A) and passing through the circle of intersection, real or imaginary, of the sphere (L) and the plane BCD; let (B′), (C′), (D′) be the analogous spheres relative to the spheres (B), (C), (D).

Porism. The radical center M of the four spheres (A′), (B′), (C′), (D′) is the harmonic pole, for the tetrahedron (T) = ABCD, of the radical plane μ of the spheres (L) and (R).

Let the radical plane μ meet the edges DA, DB, DC, BC, CA, AB of the tetrahedron (T) in the points U, V, W, X, Y, Z, respectively, and let U′, . . . , Z′ be the harmonic conjugates of U, . . . , Z for the respective pairs of vertices of (T).

The lines U′X′, V′Y′, W′Z′ have a point, say M, in common which is the harmonic pole of the plane μ = UVWXYZ for the tetrahedron (T)[1]. Remains to be shown that M coincides with the radical center of the four spheres (A′), (B′), (C′), (D′).

The planes BCD, ACD are respectively the radical planes of the pairs of spheres (A′) and (L), (B′) and (L), hence the radical plane of (A′) and (B′) passes through the line CDWW′. Again, the planes μ, BCD are the radical planes of the pairs of spheres (L) and (R), (L) and (A′), hence the radical plane of the spheres (R) and (A′) passes through the line XVW = (μ, BCD). Now both (A′) and (R) are orthogonal to the sphere (A), hence the center A of (A) lies in the radical plane of (A′) and (R), and therefore the radical plane of (A′) and (R) coincides with the plane A-XVW. Similarly the radical plane of (B′) and (R) coincides with the plane B-YUW. Consequently the radical plane λ of (A′) and (B′) passes through the line of intersection of the planes A-XVW, B-YUW.

It follows immediately that the point W belongs to the radical plane λ of (A′) and (B′), as has been noticed already. Moreover, the line AV of the plane A-XVW and the line BU of the plane B-YUW

both lie in the plane ABD, hence they have a point, say K, in common, and K belongs to the plane λ. Now the line DK meets AB in the point Z′, and both K and D belong to λ, hence Z′ also lies in λ. Thus the radical plane λ of the spheres (A′), (B′) passes through the points Z′ and W′, hence λ passes through the point M.

Similarly for the radical plane of any other two of the four spheres (A′), (B′), (C′), (D′). Consequently the radical center of these four spheres coincides with M, which proves the proposition.

The above porism gives rise to a large number of propositions. Some examples will be given in what follows.

2. We may take for (L) the circumsphere (O) of the tetrahedron (T) = ABCD, and for (A′), (B′), (C′), (D′) the spheres having for great circles the circumcircles of the triangles BCD, CDA, DAB, ABC, respectively. The radical center of these four spheres is the isogonal conjugate, with respect to (T), of the circumcenter O of (T)[2].

Then (A), (B), (C), (D) are the spheres having A, B, C, D for centers and orthogonal respectively to the spheres (A′), (B′), (C′), (D′). The radical plane of the orthogonal sphere (R) of (A), (B), (C), (D) with the circumsphere (O) = (L) of (T) is the newtonian plane of the four spheres.[3] We thus have the proposition:

If with the vertices of a tetrahedron (T) *as centers spheres* (A), (B), (C), (D) *are drawn respectively orthogonal to the spheres having the circumcircles of the faces of* (T) *for diametral circles, the newtonian plane of the spheres* (A), (B), (C), (D) *has for harmonic pole with respect to* (T) *the isogonal conjugate, for* (T), *of the circumcenter of* (T).

3. a. Suppose we take for (A′), (B′), (C′), (D′) the spheres passing through a given point M and through the circles of intersection of the faces of a tetrahedron (T) = ABCD with the given sphere (L). The point M is the radical center of the four spheres passing through M, and the harmonic plane of M for (T) will be the radical plane of (L) and the orthogonal sphere (R) of the four spheres (A), (B), (C), (D) having A, B, C, D for centers and orthogonal respectively to (A′), (B′), (C′), (D′).

b. Suppose further that M coincides with the centroid G of (T), and that (L) is concentric with the circumsphere (O) of T.

The radical plane of (R) and (L) is the harmonic plane of G for (T), i. e., the plane at infinity, consequently the spheres (R) and (L) are concentric. Thus the orthogonal sphere (R of (A), (B), (C), (D)

is concentric with the circumsphere (O) of (T), i. e., the center of (R) is the circumcenter O of (T).

Let a, b, c, d, r, be the radii of the spheres (A), (B), (C), (D) (R). We have:

$$a^2+r^2=OA^2, \cdots, \cdots, d^2+r^2=OD^2;$$

but OA = OB = OC = OD, hence $a=b=c=d$.

Thus: *The powers of the vertices of a tetrahedron (T) with respect to the spheres determined by the centroid of (T) and the circles of intersection of the respectively opposite faces with a sphere (L), of arbitrary radius, concentric with the circumsphere of (T), are equal.*

c. If we assume that (L) coincides with the circumsphere (O) of (T), the above proposition may be stated as follows: The powers of the vertices of a tetrahedron with respect to the spheres determined by the centroid and the remaining three vertices, are equal[4].

4. May we remind the reader that the six points of intersection of the six edges of a tetrahedron (T) with the external bisecting planes of the respectively opposite dihedral angles of (T) lie in a plane.[5] We shall denote that plane by γ. The harmonic pole of γ for (T) is the in-center I of (T).[6]

Now suppose we take for (A), (B), (C), (D) the spheres having for centers the vertices of a tetrahedron (T) = ABCD and passing respectively through the points of contact A′, B′, C′, D′ of the opposite face of (T) with the inscribed sphere (I) of (T).

Suppose further that we take for (L) the sphere (I). The spheres (A′), (B′), (C′), (D′) thus reduce to the points A′, B′, C′, D′. The orthogonal sphere of the four point spheres thus coincides with (I), hence their radical center coincides with the center I of (I). Consequently the harmonic plane γ of I for (T) is the radical plane of the sphere (I) = (L) with the orthogonal sphere (R) of the four spheres (A), (B), (C), (D); hence γ is perpendicular to the line joining the centers R, I of the spheres (R), (I). Thus:

The line joining the incenter of a tetrahedron (T) to the radical center of the four spheres having for centers the vertices of (T) and passing through the points of contact of the respectively opposite face of (T) with its inscribed sphere is perpendicular to the plane γ of (T).

Analogous propositions may be formulated for the other quadritangent spheres of (T).

5. **Suppose we take four spheres (A), (B), (C), (D) with radii equal to zero. Their orthogonal sphere (R) coincides with the circumsphere (O) of the tetrahedron (T) = ABCD formed by their centers, and our porism becomes:**

The radical center of the four spheres determined by the vertices of a tetrahedron (T) and the circles of intersection of the respectively opposite faces of (T) with a given sphere (L), is the harmonic pole, for (T), of the radical plane of (L) and the circumsphere (O) of (T).

This proposition is due to the Belgian mathematician J. Neuberg (1840 - 1926), co-founder of the modern theory of the triangle and the tetrahedron.[7]

References

1. Nathan Altshiller-Court, *Modern pure solid geometry*, pp. 230. ff. Macmillan, 1935. This book will be referred to as MPSG.

2. MPSG, p. 245, art. 753.
3. „ p. 203.
4. *National Mathematics Magazine*, vol. 15 (1941), p. 274.
5. MPSG, p. 72, art. 238.
6. MPSG, p. 239, art. 735.
7. *Mathesis*, 1922, p. 406.

GLEANINGS

The fundamental distinction between inspired music, chess and mathematics on the one hand and professions, industry and politics on the other is that while the former manifests in the subject without the subject's volition the latter is the direct product of the will and desire of the person concerned, Ramanujam did not desire to become a mathematics prodigy; nor did, Reshevsky or to come near home, our Balachandran exert himself to become a chess genius. But Henry Ford and F. D. Roosevelt are the products of a desire and will to forge ahead in a practical world with all that the process implies. From this it is but one step to infer that while inspired music, mathematics and chess descend upon man—the incarnated soul—from extra-mundane orders of Reality, professions, politics and industry are gathered from the world by the man successful in these respectively. Viewed in this light it is a left-handed compliment to the native chess genius of the Cuban master to say that had he but diverted his attention to 'practical' spheres of life, he would have shone as well as he did in chess.

T. A. KRISHNAMACHARIAR in *The Hindu.*

An Indian Philosopher, being asked what were, according to his opinion, the two most beautiful things in the Universe answered: The starry heavens above our heads, and the feeling of duty in our hearts.

BOSSUET.

ON SOME PROBLEMS OF TRANSPORT ECONOMICS

BY

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Here we deal with passenger transport between two consecutive termini, the fares being "fixed". In the first section, we ignore the time factor and suppose the fare p is constant. In the next section we deal with the case in which the fare is a step-function of the time $p(t)$.[1]

We suppose that to each fare or fare-function, there is correlated a passenger offering function representing the number of passengers wishing to travel at this fare. In the first section, we treat this as a number $f(p)$ while in the second section we consider it as a distribution in time. The permissible fares are the integral multiples of the least unit of currency and the passenger offering function takes integral or, if we consider averages, rational values.[2] The problem, so limited, is of little interest mathematically. We therefore assume that p can vary continuously. We suppose, also that the passenger offering function has a suitable degree of "smoothness" and derivatives up to the second order, and further that the transport conditions reproduce themselves exactly after a certain period T and we use the parameter t to express the position of an instant in this period.[3] We also assume the existence of a cost function which is also a step-function. The operator is supposed to adjust the variables at his command so as to maximize his profit or the difference between

[1] Vide R. G. O. Allen. *Mathematical analysis for Economists* page 130 Ex. 10 and 11. This paper may be said to deal with the so called Dynamical Problem (*c. f.* for instance J. R. Hicks "Wages and interest, The Dynamical Problem." *Economic Journal* Vol. XLV Sept. 1935.) inasmuch as our problem involves time. It is however characteristic of transport problems that the cost function depends on the distribution of traffic in time. We cannot, also, consider "staggered" fares without considering the time factor. In this sense a static discussion cannot be realistic.

[2] *c. f.* Allen *loc. cit.* pg. 110-112. The range of values is further extended if one supposes that the real value of the currency unit varies with time.

[3] In a static investigation of a long term process, we really assume that an instantaneous section in time is completely representative and that we need consider no factors outside this section our assumption of perfect periodicity means than any section in time of length T is exactly similar to any other section. Our problem is, in this sense, intermediate between the static and dynamic problems, in which no such assumption is made.

the product of the fare and the passenger offering (or in the second section the corresponding integral) and the cost function.

1. In this section we neglect the time factor and suppose that the cost function R depends only on the number of passengers x transported but that $R(x)$ may be discontinuous in X at an enumerable set. We deal with the case of a single operator or monopolist. The profit $P = px - R(x) = p \cdot f(p) - R\{f(p)\}$.

The only difficulty in maximizing this function is that R may have discontinuities and the normal method is therefore not applicable. With a known $f(p)+R$ all we have to do is to compare the maxima, found in the usual way[1] in the enumerable set of continuous stretches, with the values at the ennumerable set of discontinuities. If this operator runs in competition with some other form of transport with fares fixed well in advance, a say, the same reasoning applies with the parametric function $f_a(p)$ replacing $f(p)$.

2. We again consider a monopolist but we no longer suppose p a constant but a step-function of the time t. We metricize the space of the step-functions p by assigning the distance-function

$$\rho\{p_1(t), p_2(t)\} = \int_0^T |p_1(t) - p_2(t)| \cdot dt.$$

It is clear that this function satisfies the three requirements of a distance function.[2] We also consider two capacity functions $N(t)$ and $N'(t)$ for the two termini. These functions specify the time of departure of the buses from the termini (we are mainly concerned with the times of departure as we shall see) and the seating capacity of the departing vehicles. In the interval between consecutive departures, we suppose $N(t)$ and $N'(t)$ to be equal to their values at the next departure, thus making the two functions step-functions. We metricize these two spaces in the same way as for $p(t)$.[3] For the composite space $\{p(t), N(t), N'(t)\}$ we take the square root of the sums of the squares of the distances between the co-ordinates.

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[1] *c. f.* Allen *loc. cit.* § 8. 7.

[2] *c. f.* C. Kuratowski *Topologie I* pp. 82, 83; or Sierpinski: *General Topology* chap. VI. If we restrict the values of p to rational values, it is clear that the number of such functions is $\leqslant c^r = c$ where c is the number of points in a linear continuum, and r the cardinal number of an enumerable aggregate. On the other hand by considering the step-functions $p(x)=0,\ x < a,\ p(x)=1\ x \geqslant a$, we see that there are certainly c such functions. There are thus c step-function taking rational values.

[3] A similar argument shows that there are c functions $N(t)$, $N'(t)$, $f(t)$ and $f'(t)$.

We denote by $f(t)$ and $f'(t)$ the two functionals depending on $p(t)$, $N(t)$ and $N'(t)$ which represent the passenger offering in the two directions.[1]

The cost function $R(t)$ is assumed to be a functional of the $N(t)$ and $N'(t)$ only.

The gross takings are

$$\sum_{1}^{n} p(t_r)\cdot\left[\min\left\{\int_{t_{r-1}}^{t_r} f(t)\,dt\right\},\ N(t_r)\right]$$

$$+\sum_{1}^{m} p(t'_r)\left[\min\left\{\int_{t'_{r-1}}^{t'_r} f'(t)\,dt\right\},\ N'(t'_r)\right]$$

where $t_1, t_2 \cdots t_n$ and $t'_1, t'_2 \cdots t'_m$ are the times of departure at the two termini.[2]

The profit is the difference between the sum and $R(N, N')$. To find the maximum of this we have to use variational methods,[3] the $p(t)$ and the two capacity functions being the adjustable variables.

The method can be extended to case where there are several different fare functions for different categories of passengers and more than one stage. Our new variable step-functions will be the set

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[1] We work with a passenger offering function. This causes difficulties in "smoothing" and also in allowing for excess passengers left behind due to overcrowding of a departing bus. These difficulties are avoided if we consider functions $F(t)$, $F'(t)$ representing the number of persons awaiting conveyance in the two directions at the time t we need no longer suppose these continuous.

We could also replace our $N(t)$, $N'(t)$ by total capacity functions $M(t)$, $M'(t)$ step-functions representing the total seating capacity of buses leaving in each direction up to but excluding t. The expression for the gross takings can then be written

$$\int_0^T p(t)\,[F(t)]_{M(t+0)-M(t-0)}dM + \int_0^T p(t)\,[F'(t)]_{M'(t+0)-M'(t-0)}dM$$

where the square bracket indicates that the function is limited to $M(t+0)-M(t-0)=N(t)$ (*c. f.* de la Vallée Poussin *Integrales de Lebesgue, Fonctions d'ensemble, classes de Baire* 1916 ed. p. 45).

This integral need not exist in the Riemann sense as $p(t)$, $F(t)$ and $M(t)$ can have common discontinuities but it must exist in the Moore-Pollard sense (not generalized) with suitable conventions regarding the sense of the discontinuities of $p(t)$, $F(t)$ and $M(t)$. *c. f.* L. C. Young. An inequality of the Holder type, connected with Stieltjes Integration *Acta Mathematica* Vol. 67 pg. 263 and Love and Young Fonctionelles Linéairies *Fundementae Mathematica*. Tome XXVIII.

[2] We could easily adjust our method to the case where different fares are charged in the two directions.

[3] *c. f.* Allen *loc. cit.* chap. 20. De la Vallée Poussin *Cours d'Analyse* chap. X See also *Methodes Topologiques dans les Problemes variationnels* by L. Tusternik and L. Schnirelmann.

We use our metric merely to define a neighbourhood of our maximizing step-function. Any metric will give the same result as our problem requires an absolute and not merely relative maximum.

of the corresponding $\{p(t)\}$ for the categories and stages and the set of capacity functions for the various possible combinations of stages.

The adjustment of the method to the case of competition by the Railway is obvious.

We now consider the case[1] of n operators plying between the same two consecutive termini. For simplicity, we limit ourselves to the case analogous to section, in which the time factor can be left out of consideration. We suppose the existing fares have been distributed and we wish to determine the new fares after the period of readjustment. We denote by $x_1, x_2 \cdots x_n$ the number of passengers transported by each and suppose that

$$x_r = f_r(p_1, p_2 \cdots p_r \cdots p_n) \qquad (r = 1 \cdots\cdots n.)$$

Let $R_1(x_1) \cdots\cdots R_n(x_n)$ be the corresponding cost of operation. Each transporter seeks to select that p_r which will maximize his profits but his decision will depend on the fares which he supposes the others will charge. Let us denote by $p_j{}'$ the fare which the rth operator thinks the jth operator will charge. Assigning these values to the $p_1, p_2 \cdots p_{r-1}, p_{r+1} \cdots p_n$ in f_r, we could determine the value of p_r which would maximize the rth operator's profit by the methods of §1. Let us suppose this method gives us the n equations

$$\phi_r(p_1{}', p_2{}' \cdots p_r \cdots p'_n) = 0 \qquad (r = 1, 2 \cdots n) \qquad \ldots \quad \text{(A)}$$

For equilibrium a necessary condition is that the final fares $p_1, p_2 \cdots p_n$ satisfy the set of equations (A) for otherwise one or more operators would increase his or their profit by changing his fare. This gives us n equations to determine these n quantities.[2]

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[1] *c. f.* Allen, *loc. cit.* § § 8·8 and 13 9 for the duopoly problem treated on the basis of "conjectural variations." It does not, however, deal with the further reactions of the duopolist when he finds his conjecture to be incorrect.

[2] If we suppose the "ϕ"s refer to the relations obtained in section 2, the same relations remain formally correct when the "p"s are considered as step-functions.

A DIALOGUE *

BY

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Student. The problem enunciated at the beginning of this chapter seems to be a very easy one, but I have seen such words as "vectorspace," "rank," "matrix" later in the book; I also noticed determinants and formulas with upper and lower indices. I cannot understand why the author is trying to make a very simple thing so complicated. The problem can be solved with the help of methods which I learned, when I read for the matriculation examination.

Tutor. Of course, it is my duty to help you to understand this theory clearly, but Mathematics is not a matter of seniority. History shows several examples where mathematicians were superior to their masters at a very early age of life. I should not miss the opportunity to learn something from you; please, explain your solution of the problem.

St. It is simply the method of substitution; From the last equation it follows that $x=(k-k_1 x_1-\cdot\cdot\cdot-k_{n-1} x_{n-1})\div k_n$. Putting this value into the remaining linear equations, I get linear equations with $n-1$ unknown only. After having solved this system, we calculate the value of x_n by putting the values of $x_1, \cdot\cdot\cdot, x_{n-1}$ in that equation. Is it so?

T. Yes — provided $k_n \neq 0$.

St. In the case $k_n=0$, x_n is infinite.

T. I do not think so;—*e g.* Consider two equations and $n=2$, $k_2=0$, say $x_1+2x_2=5$, $x_1+0x_2=1$.

The only solution of this system is obviously $x_1=1$, $x_2=2$.

St. Yes, that is true.—If $k_n=0$, then I take anotber equation of the system in place of the last one. Thus without loss of generality, I suppose $k_n \neq 0$. I think you will be satisfied.

T. Unless the coefficient of x_n is zero in each equation of the system.

* [The Socratic method of eliciting hidden difficulties by means of a dialogue is pleasant and profitable, and of considerable pedagogic interest. We have great pleasure in reproducing the following dialogue from Prof. Levi's recent publication "Algebra, vol. I" with the kind permission of the Calcutta University Authorities and of the Author—*Ed.*]

St. In this case the system has to be considered as a system with $(n-1)$ variables only; it would be absurd to consider it as a system of n variables as the equations are actually independent of x_n.

T. Perhaps less absurd than you may believe, but I accept your definition that a system of linear equations should be considered to depend on such variables only, as have at least one coefficient different from zero.

St. Certainly.

T. After x_n has been eliminated, how do you continue?

St. I shall repeat the process again and again until I get one equation with one variable x_1, and then there is no problem left.

T. You suppose that the number of equations is equal to the number of variables, and you believe that at every step of your procedure, *both the numbers decrease by exactly one?*

St. Certainly, but the number of the equations may also be less than the number of the variables, let us say $m < n$ equations in n variables. In this case one puts the term with $x_{m+1}, \cdots, x_n$ to the right hand side. These variables may take arbitrary values. For every set of values $x_{m+1} \cdots, x_n$, there exists one solution $x_1, \cdots, x_r$ as there are as many of these variables, as there are equations. The number $n-m$ is the *degree of freedom* of our system as the values of $n-m$ variables may be chosen arbitrarily.

T. And if there are more equation than variables?

St. Then there cannot exist any solution. It is obvious that n variables cannot satisfy a system of more than n conditions.

T. But it seems to me that the system $x=1$, $2x=2$ has a solution although it is a system of two equations with one variable.

St. But these equations are not different. Equations which differ by a common factor only cannot be considered as different, and it is commonsense to consider only such equations which are different.

T. Thus, if two equal equations are given, one of them should be dropped.

St. Yes.

T. But this must be done also at the later stages of the procedure,

St. I cannot follow you exactly.

T. Consider:
$$7x_1 - 38x_2 + 3x_3 = 13$$
$$3x_1 + 13x_2 + 2x_3 = 17$$
$$2x_1 - \ 5x_2 + \ x_3 = 6$$

3 different equations in x_1, x_2, x_3. Since the degree of freedom is zero, do you expect to get exactly one solution?

St. Yes; Put $x_3 = 6 - 2x_1 + 5x_2$ in the first two equations then
$$x_1 - 23x_2 = -5$$
$$-x_1 + 23x_2 = \ 5.$$

T. These equations differ by a factor -1 only, hence one of them must be dropped. Thus you may chose x_2 in an arbitrary manner. Put $x_1 = 23x_2 - 5$. $x_3 = -41x_2 + 16$, and this will solve the *system of equations for every value of x_2 You have one "degree* of freedom," although the number of the equations is equal to the number of the variables.

St. That is true. This example is obviously a wicked exception.

T. You may call it an exception if you like, but there are a plenty of them.

St. I see;—There may be a certain cases, where the degree of freedom is higher than the difference between the number of the variables and the number of the equations, but at any rate n equations with n variables have at least one solution which can be found by the method of substitution.

T. Why?

St. The number of the equations can decrease, as one may get two equations by the procedure of substitution, and then one of them must be dropped, but the number of the variables cannot.

T. Try:
$$9x_1 - 15x_2 - 3x_3 = 13$$
$$3x_1 + 10x_2 + 2x_3 = \ 1$$
$$2x_1 - \ 5x_2 - \ x_3 = \ 2.$$

St. Substitute $x_3 = 2x_1 - 5x_2 - 2$ in the first and the second equation.

Hence $3x_1 = 7$

$7x_1 = 5.$ This is funny.

T. Indeed, the coefficients of x_2 in both equations become zero, thus the equations have to be considered as equations of one variable only. Now x_1 should be equal to $7:3$ and to $5:7$, that is impossible.

St. Perhaps I was somewhat rash in conceding that a system of equations in n variables should be considered as system in $(n-1)$ variables if the coefficient of one of the variables are all equal to zero. Let us retain x_2 and put $3x_1 - 0x_2 = 7$
$7x_1 + 0x_2 = 5$. Hence $x_1 = 5/7 - 0x_2$, and therefore $0x_2 = 34:7$, $x = \infty$.

T. What do you mean by ∞?

St. Infinity; that is a number which is greater than every other number and equal to $1:0$.

T. Can you calculate with this ∞ as with an ordinary number?

St. Certainly.

T. Then $-\infty = -1:0 = 1:(-0)$ and since $-0=0$, $-\infty = \infty$ holds. Hence $0 = 2\infty$, and therefore $0 = \infty$.

St. No, that is not so. One cannot calculate with this symbol as with an ordinary number. But, as a matter of fact, this ∞ occurs in mathematics. It is somewhat complicated matter, one needs differential calcus to handle it, and I was hoping that you may explain it to me clearly one day.

T. On another occasion. The symbol ∞ does occur; sometimes it is used rightly, sometimes wrongly. Use and misuse, both are found in text books. Considering the systems of linear equations, we enquire about those numbers which taken for $x_1, \cdots, x_n$ respectively, satisfy those equations. Numbers can be added, subtracted and multiplied; one can also divide a number by a number, unless the divisor is zero. The division by zero is meaningless as far as numbers are concerned.

St. And the example which I attempted just before?

T. It has no solution, since x_1 cannot simultaneously be equal to $7:3$ and equal to $5:7$.

St. So there exist systems of 3 equations in 3 variables which have no solutions, systems which have an infinity of solutions, and systems which have exactly one solution. How did you construct those examples by which you cornered me.

T. It is not difficult if one knows a little bit of theory.

St. Those vectorspaces, matrices, rank etc.?—Sir, I should be thankful if you could explain to me some portion without using those notions. I do not like those innovations.

T. Then try the simplest case; $ax=b$.

St. Then $x=b:a$.

T. Provided $a \neq 0$.

St. If $a=0$, $b \neq 0$, the equation has no solution as there exists no number x, for which $0x=b \neq 0$. If however $a=0$, $b=0$, then every value x is a solution.

T. Indeed; This simple case is the seed of the whole theory.

Now try. $$a_1x+a_2y=a$$ $$b_1x+b_2y=b.$$

St. $\Delta y=\Delta_1$, $\Delta x=\Delta_2$, where $\Delta=a_1b_2-b_1a_2$, $\Delta=a_1b-b_1a$, $\Delta_2=ab_2-ba_2$. If $\Delta \neq 0$, then $x=\Delta_2:\Delta$, $y=\Delta_1:\Delta$.

T. In this case there exists no other solutions, and these values satisfy the given equations, as you may verify easily.

St. If $\Delta=0$, but Δ_1 or Δ_2 is different from zero, there is no solution. If $\Delta=\Delta_1=\Delta_2=0$, then every couple of values (x, y) satisfies the equations.

T. Consider $x+2y=5$, $3x+6y=15$. Here $\Delta=\Delta_1=\Delta_2=0$, but *e.g.* $(x, y)=(0, 0)$ is not a solution, as $x=5-2y$.

St. This is true, but I cannot understand it. The equation $\Delta x=\Delta_2$, $\Delta y=\Delta_1$ are satisfied by every pair of values x, y if $\Delta=\Delta_1=\Delta_2=0$'

T. These equations are consequences of the given equations, they are *necessary* conditions for solutions x, y of the given systems but they may not be *sufficient* ones. The case $\Delta=\Delta_1=\Delta_2=0$ contains different cases.

(1) If all the coefficients are equal to zero, then every pair (x, y) is a solution.

(2) Let $a_1=a_2=b_1=b_2=0$, but $(a, b) \neq (0, 0)$ then there is no solution.

(3) Let at least one of the coefficients on the left side be$\neq 0$. Without loss of generality, $a_1 \neq 0$, Put $b_1 : a_1 = \lambda$.

Hence, $0 = \Delta = a_1(b_2 - \lambda a_2),\ b_2 = \lambda a_2$
$0 = \Delta_1 = a_1(b - \lambda a),\ b = \lambda a.$

$(b_1 x + b_2 y - b) = \lambda(a_1 x + a_2 y - a)$. Hence $x = (a - a_2 y) : a_1$ for arbitrary y furnishes all the solutions. There are therefore five different cases.

St. For a higher number of variables and of equations a full analysis may become very complicated. How to tackle the problem for an arbitrary n?

T. For this, I propose to you to study the notions of vector-space, rank, matrix, determinant etc., as explained in the following articles.

GLEANINGS

Marcellus, though at a loss what to do, and not knowing how to oppose the machines of Archimedes, could not, however, forbear jesting upon them. "Shall we persist," said he to his workmen and engineers, "in making war against this Briareus of a geometrician, who treats my galleys and sambucas so rudely? He infinitely exceeds the fabled giants, with their hundred hands, in his perpetual and surprising discharges upon us." Marcellus had reason for complaining of Archimedes alone. For the Syracusans were really no more than members of the engines and machines of that great geometrician, who was himself the soul of all their powers and operations. All other arms were unemployed; for the city, at that time, made use of none, either defensive or offensive, but those of Archimedes. Marcellus at length, perceiving the Romans so much intimidated, that if they saw upon the walls only a small cord, or the least piece of wood, they would immediately fly, crying out that Archimedes was going to discharge some dreadful machine upon them, renounced his hopes of being able to make a breach in the place, gave over his attacks, and turned the siege into a blockade. * * * Deprive Syracuse of only one old man, the great strength of the Roman arms must take the city.

From ROLLIN: *History of the Ancients.*

ANOTHER DIALOGUE †

By

Prof. F. W. Levi, *Calcutta University.*

Student. When I started reading Algebra, you advised me to study carefully the systems of linear equations. So I did. I further read general Algebra and continued fractions. At first it was hard work, but later on, I was quite successful.

Tutor. Alright, but I do not think you have met me for this. You are looking rather despondent.

Stu. Indeed, I am again in the wilderness.

Tu. Why, is there some difficulty in the book which you want me to explain?

Stu. It is not for that, but the whole subject became problematical to me again.

Tu. I wonder how.

Stu. Yesterday an engineer asked me to solve a certain algebraic equation. I replied that, in consequence of the fundamental theorem of General Algebra, there exist roots in a suitable extension of the field of the coefficients, and that from the fundamental theorem of classical Algebra, this extension can be chosen as the field of the complex numbers. Thus there exist *complex roots of the equation* and some of them might be real.

Tu. Was the engineer satisfied by your reply?

Stu. Not at all; he said that I seemed to be a great philosopher, and that I had missed the point completely. He was interested in real roots only. And he had no doubt about their existence. He has found out that the force (expressed in kilogrammes) acting on a certain part of an engine was bound to satisfy that equation. He was asking me to compute that force and nothing else.

Tu. And you could not. The polynomial was too complicated.

† [This is also from Prof. Levi's *Algebra* and is published with the permission of the author and the Calcutta University Authorities.—Ed.]

Stu. It looked very simple. It was something like this x^5+4x^3+2x+6. From Eisenstein's theorem it is irreducible and therefore its real roots must be irrational. This I told the engineer.

Tu. Perhaps the good man did not know anything about irrationality.

Stu. He did! But he was not at all interested in my statement. He said, "I do not want to have an infinity of decimals, even if you can provide me with them. Compute the kilogrammes; I leave the grams etc. to you." Now, for any positive x the polynomial takes positive values only. So, I told him that the real roots of the polynomial must be negative.

Tu. And, was this statement of any use to the engineer?

Stu. No, he knew already that the force was directed to the negative side; and then he said. 'The direction of the force is not very interesting to me, as there is little difference whether the material is exposed to stress or to pressure. If you give me a solution with 30% of error and a wrong sign, I could make some use of it. But your philosophical talk is worth nothing." He was quite rude eventually.

Tu. And you?

Stu. I am bewildered. After having read about 200 pages of the book, I am still unable to solve a very simple algebraic problem, not even if 30% of error and a wrong sign are admitted. Though I got very interested in Algebra, the Engineer's argument has impressed me. I am afraid that all my hard work has seen spent uselessly.

Tu. I rather think, you have stopped reading at the wrong place. If you continue, you will be able to provide your friend with a solution, with considerably less than 30% of error.

Stu. I had already a glance on the next chapter but I do not see any connection between its content and the preceeding parts of the book (eg.) general Algebra; and then, there is another thing that strikes me, every solution is given only approximately. I should like to know the solutions correctly. If for a particular application, a few decimal places only are requested, then I may neglect the higher terms of the correct result. But as a student of Pure Mathematics, I must know at first the proper solution, before admitting some error for the sake of abbreviation.

Tu. How do you want to represent the solution if it happens to be an irrational number?

Stu. There are many ways of expressing irrational numbers. For instance $\sqrt{2}$ is irrational. It cannot be expressed as a ratio of 2 integers, but nevertheless, $\sqrt{2}$ is a number. Everybody knows what is $\sqrt{2}$.

Tu. Suppose that I do not know and you try to explain.

Stu. $\sqrt{2}$ is the positive number whose square is equal to 2.

Tu. Well, I take it for granted that one and only one such positive number exists. Let x be positive and $x^2=2$. Then $x=\sqrt{2}$, or $\sqrt{2}$ is the positive root of $x^2-2=0$. I think this statement is completely equivalent to yours.

Stu. It is.

Tu. You seem to be satisfied with this manner of expressing irrational numbers.

Stu. Of course, I am. If I could express the roots of every algebraic equation in this way, there would be nothing to complain of.

Tu. My point is that in this case, the roots are expressed by a tautology.

Stu. I cannot follow you.

Tu. Listen, which are the roots of the polynomial x^2-2?

Stu. $\sqrt{2}$ and $-\sqrt{2}$.

Tu. Whereby $\sqrt{2}$ is nothing else than a symbol for the positive root of x^2-2. Besides the statement, that x^2-2 has two real roots and that their sum is equal to zero, your solution of the problem to find the roots of x^2-2 is a mere tautology. Your conclusion goes like this: "Who is Amal?" "The Brother of Bimal"— "And who is Bimal?" "Amals' brother." That means only that there exist two brothers Amal and Bimal, but it does not explain who is Amal.

Stu. But $\sqrt{2}$ is a well-known number; Mathematicians have got used to it and they calculate with $\sqrt{2}$ as they do with 23 or 1 : 7. For me, there is no problem about $\sqrt{2}$.

Tu. Is it for the symbol $\surd$, that you hold this opinion?

Stu. $\surd$ as a symbol is a mere convention, the mathematicians could use any other notation instead of it, but I do not see any reason why symbols familiar to everybody should be replaced by new ones.

Tu. I fully agree with you, but for the sake of our conversation let us denote the real roots of a polynomial $f(x)$, as far as they exist, by $[f(x)]_1$, $[f(x)]_2$, $\cdots$ $[f(x)]_n$ in their order of magnitude starting with the greatest root.

Then your explanation of $\surd 2$ simply means that $\surd 2 = [x^2 - 2]_1$.

Stu. Now you want me to admit that for every $f(x)$ which has a real root, the symbol $[f(x)]_1$ must be considered as a solution of the equation $f(x) = 0$. You propose that there is no higher justification in considerating $\surd 2$ as the given number than e. g. $[x^6 + 4x^5 + 2x + 6]_1$. But there is a huge difference between these two cases.

Tu. How is that?

Stu. We know more about $\surd 2$ than that it is positive and that its square is equal to two.

Tu. What do you know about $\surd 2$?

Stu. $\surd 2 = 1{\cdot}4142 \cdots$

Tu. $1{\cdot}4142$ is a rational number, whereas $\surd 2$ is irrational.

Stu. Certainly, it is an infinite decimal fraction, but they have computed 200 decimals or even more of them. You cannot deny that $\surd 2$ is very well known.

Tu. There still remains a certain error.

Stu. But a negligible one.

Tu. That depends on the purpose of the calulation. I was told that certain students of Pure Mathematics must know the proper solution before admitting some error for the sake of abbreviation, was it not so?

Stu. But $\surd 2$ is uniquely determined as the only positive root of $x^2 - 2$.

Tu. Yes there exists one and only one such root. This is a statement on existence and on uniqueness but nothing more than that.

I think we have agreed already about this item. On the other hand, I admit that we know more than that about $\sqrt{2}$. For instance approximately equal to $1{\cdot}4142 \cdots$. Or to put it more clearly, $\sqrt{2}$ lies between $1{\cdot}4142$ and $1{\cdot}4143$. One can find out easily small intervals where $\sqrt{2}$ is situated. There is no limit to the improvement of the approximation, and the diminution of the error. This error is not a kind of mistake which is the result of a negligent treatment. On the contrary it is an essential part of the solution of the problem. One cannot determine irrational numbers otherwise than approximately. This fact is concealed by symbols we are using. Numbers represented by them are uniquely determined in the sense that there exists one and only such number but one cannot determine the place of an irrational number on the real axis otherwise than approximately.

Stu. Thus a formula like $\sqrt[3]{14+\sqrt{170}}$ is only a recipe how to determine an irrational number approximately.

Tu. The formula denotes the greatest root of $x^6 - 28\,x^3 + 26$ and it shows of course a recipe how to compute that number approximately. A mental calculation furnishes 3 as a first approximation which would satisfy your friend completely.

Stu. Suppose one could represent every root of a polynomial by the help of similar symbols. This would furnish a recipe to determine every root approximately.

Tu. As a matter of fact, not every root is representable in that manner, and even if it is, one prefers a different method sometimes.

Stu. My impression is that those methods have no connection with General Algebra. Theory of Approximation and General Algebra apparently belong to different branches of Mathematics if not to two different sciences.

Tu. They are complementary to each other. You already mentioned the two fundamental theorems which state the existence of roots in certain fields. They must be supplemented by investigations about where the roots are situated in the field. The methods of investigation must tally with the structure of the particular field under consideration; they cannot be of a general nature. The real numbers are ordered linearly, whereas the complex numbers correspond to the points of a plane. Hence one subdivides the real axis into intervals to determine the real numbers and

similarly the plane is subdivided into certain domains [eg. rectangular or circular ones] to locate complex numbers. Both ways lead to an approximate determination of number (eg.) roots of a polynomial.

Stu. As the n roots of the polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ are uniquely determined by the numbers $a_0 \cdots a_{n-1}$, there must exist functions $f(a_0 \cdots a_{n-1})$ which show the distribution of roots in the complex plane. One should investigate these functions; this would be a worthy continuation of General Algebra.

Tu. If you take the word function in the most general sense, there exist indeed such functions $f(a_0 \cdots a_{n-1})$ *e. g* the set of the roots itself is one. The problem is how to represent these functions; you should not expect that all of them are polynomials in $a_0 \cdots a_{n-1}$. Every polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ can be represented by a point $P = (a_0 \cdots a_{n-1})$ of an n-dimensional space. There are theorems stating that if certain inequalities in $(a_0 \cdots a_{n-1})$ hold *i. e.*, if P is situated in a particular domain of the n-dimensional space, the n roots are distributed in the complex plane in a particular manner. These investigations are very interesting, but at the present time, the approximation of the roots is based more on the methods of calculation than on these theorems. In many cases, the theorems seem to be the result of the practice of calculation. For this reason, the author has started from Horner's scheme which gives the clue to the whole theory. I advise you to work out many numerical examples, it will help you to understand the theoretical portion!

DEATH OF ARCHIMEDES

Archimedes, at a time when all things were in this confusion at Syracuse, shut up in his closet like a man of another world, who has no regard for what is passing in this, was intent upon the study of some geometrical figure, and not only his eyes but the whole faculties of his soul were so engaged in this contemplation that he had neither heard the tumult of the Romans universally busy in plundering, nor the report of the city's being taken. A soldier, on a sudden, comes in upon him and bids him follow him to Marcellus. Archimedes desired him to stay a moment, till he had solved his problem and finished the demonstration of it. The soldier who neither cared for his problem or demonstration, enraged at this delay, drew his sword and killed him. Marcellus was exceedingly afflicted when he heard the news of his death. Not being able to restore him to life, of which he would have been very glad, he applied himself to honour his memory to the utmost of his power. He made a diligent search after all his relations, treated them with great distinction and granted them peculiar privileges. As for Archimedes, he caused his funeral to be celebrated in the most solemn manner and erected to him a monument among the great persons who had distinguished themselves most at syracuse.

From ROLLIN: *History of the Ancients*.

NOTES AND DISCUSSIONS

Metrization by means of a function satisfying a single axiom.

1. The object of this note is to show that if, for every pair of elements x, y of a set, a real-valued function $\rho(x, y)$ is defined satisfying the axiom

$$\text{(A)} \qquad \rho(x, z)+\rho(y, z) \geqslant \rho(x, y)+\rho(z, z)$$

then the set can be metrized. This is done by constructing a function $d(x,y)$ in terms of $\rho(x, y)$ which shall satisfy the metric axioms

$$\begin{array}{lll} \text{(B)} & B_1: & d(x, x)=0 \\ & B_2: & d(x, y)=d(y, x) \\ & B_3: & d(x, y)+d(y, z) \geqslant d(x, z). \end{array}$$

Lemma 1: $\rho(x, y)=\rho(y, x)$.

Proof: Putting $z=x$ in (A) we get $\rho(y, x) \geqslant \rho(x, y)$, and from symmetry $\rho(x, y) \geqslant \rho(y, x)$, whence the results follows.

Lemma 2: $\rho(y, z) \geqslant \frac{1}{2}[\rho(y, y)+\rho(z, z)]$.

This is easily proved by putting $x=y$ in (A).

2. Now set $d(x, y)=\rho(x, y)-\frac{1}{2}[\rho(x, x)+\rho(y, y)]$.

By lemma 1 it is easily seen that $d(x,y)$ satisfies axioms B_1 and B_2. That it satisfies axiom B_3 is proved as follows:

$$d(x, z)+d(z, y)=\rho(x, z)+\rho(z, y)-\rho(z, z)-\tfrac{1}{2}[\rho(x, x)+\rho(y, y)].$$

$$\geqslant \rho(x, y)-\tfrac{1}{2}[\rho(x, x)+\rho(y, y)]. \text{ by lemma 1 and axiom (A).}$$

$$=d(x, y).$$

For a metric space, in addition to the axioms B we require the axiom

$$B_4: \qquad d(x, y)>0 \text{ if } x \neq y.$$

This is achieved by strengthening lemma 2 by the axiom

$$\text{(A}'\text{)} \quad \rho(x, y)>\tfrac{1}{2}[\rho(x, x)+\rho(y, y)] \text{ if } x \neq y.$$

3. It is obvious that all distance functions satisfy axiom (A). But there are also functions which are not distance functions and which satisfy axiom (A). One trivial example is the function

$$\rho(x, y) = d(x, y) + a,$$

where d is a distance function and a is some constant. It is not easy to find many concrete illustrations of sets wherein a function ρ may be defined which is neither a distance function nor one of the trivial type given above.

The following examples may be of some interest.

Example 1. Consider the set of all real numbers and let $\rho(x,y)$ be defined to be the greater of x and y. ρ satisfies axiom (A) since

$$\max(x, z) + \max(y, z) \geqslant \max(x, y) + z,$$

The distance function defined in terms of ρ is

$$d(x, y) = \max(x, y) - \tfrac{1}{2}(x + y)$$
$$= \tfrac{1}{2}|x - y|.$$

Example 2. Consider the set of all positive integers and let the l.c.m. and g.c.d. of two numbers a, b be denoted by $[a, b]$, and (a, b) respectively. Since the equivalent relations

$$[a, c]\,[b, c] \geqslant c\,[a, b]\,; \quad \frac{1}{(a, c)(b, c)} \geqslant \frac{1}{c(a, b)}$$

always hold we may take $\rho(a,b) = \log\,[a,b]$ or $\rho(a,b) = \log\frac{1}{(a,b)}$. The ρ's so defined will both satisfy axiom (A) and they give rise to the distance function

$$d(a,b) = \log\frac{[a, b]}{\sqrt{ab}} = \log\frac{\sqrt{ab}}{(a, b)} = \tfrac{1}{2}\log\frac{[a, b]}{(a, b)}.$$

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On Conformal Transformations

Many complete criteria have been given for the conformality of a plane transformation, all amounting, of course, to the Cauchy-Riemannian conditions directly or indirectly. I add one more below, assuming that the transformation is reversible. Conformality is here viewed in terms of the parametric curves in the original and the image planes and my object is to give a property of these curves which corresponds fully to the conformal property. The assumption that the inverse transformation exists is made since with it we obtain a very simple property of the parametric curves, *viz* orthogonal intersection, as a complete criterion. As is well known, orthogonal intersection in one of the planes only is insufficient.

THEOREM : "*A necessary and sufficient condition that a reversible plane transformation* $\xi=\phi(x, y)$, $\eta=\Psi(x, y)$ *be conformal is that the transformation and its inverse be both orthogonal. i.e. that the parametric curves in the* (ξ, η) *as well as the* (x, y) *planes do intersect orthogonally.*"

The necessity of the condition follows at once from the fact that if a transformation is conformal, then the reverse transformation also is conformal and every plane conformal transformation is orthogonal.

To prove sufficiency, let the inverse transformation be

$$x=g(\xi, \eta),\ y=h(\xi, \eta)$$

so that

$$\xi=\phi\{ g(\xi, \eta), h(\xi, \eta) \}$$
$$\eta=\Psi\{ g(\xi, \eta), h(\xi, \eta) \}$$

Differentiating we obtain

$$\left.\begin{aligned} 1&=\phi_x g_\xi+\phi_y h_\xi \\ 0&=\Psi_x g_\xi+\Psi_y h_\xi \end{aligned}\right\} \qquad \begin{aligned} 0&=\phi_x g_\eta+\phi_y h_\eta \\ 1&=\Psi_x g_\eta+\Psi_y h_\eta \end{aligned}$$

which give the following relations between the partial derivatives:

(i) $\quad g_\xi=\Psi_y/\mathrm{J},\ \ g_\eta=-\phi_y/\mathrm{J},\ \ h_\xi=-\Psi_x/\mathrm{J},\ \ h_\eta=\phi_x/\mathrm{J}$

where

$$\mathrm{J}=\partial(\xi, \eta)/\partial(x, y)\neq 0$$

Now the parametric curves $g(\xi, \eta)=$ const., $h(\xi, \eta)=$ const. intersect at an angle

$$\cos^{-1}\,(h_\xi g_\xi+h_\eta g_\eta)/\sqrt{(h_\xi^2+h_\eta^2)}\sqrt{(g_\xi+g_\eta^2)}$$

and, since this is a right angle, we have

(ii) $$h_\xi g_\xi+h_\eta g_\eta=0$$

Likewise since the curves $\phi(x, y)=\text{const.}$ $\Psi(x, y)=\text{const.}$ intersect at right angles, therefore

(iii) $$\phi_x\Psi_x+\phi_y\Psi_y=0$$

In virtue of the relations (i), the relation (ii) becomes

(iv) $$\phi_x\phi_y+\Psi_x\Psi_y=0$$

Solving (iii) and (iv) we obtain

$$\phi_x=\Psi_y \text{ and } \phi_y=-\Psi_x$$

or $$\phi_x=-\Psi_y \text{ and } \phi_y=\Psi_x$$

which are the Cauchy-Riemannian equations. Hence the transformation is conformal.

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A Contact Transformation

In this paper I show that the transformation defined by the equations

(A) $$\begin{cases} X = x - \dfrac{3y''y'''}{3y''y'''' - 5y'''^2} \\ Y = y + \dfrac{3y''(3y''^2 - y'y''')}{3y''y'''' - 5y'''^2} \end{cases}$$

where the accents denote differentiations with respect to x, is a contact transformation of order four and obtain the corresponding geometrical theorem.

Let the right members of (A) be called f and ϕ respectively. Then we have

$$\frac{dY}{dX}=\frac{\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}y'+\frac{\partial\phi}{\partial y'}y''+\frac{\partial\phi}{\partial y''}y'''+\frac{\phi\partial}{\partial y'''}y''''+\frac{\partial\phi}{\partial y''''}y'''''}{\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}y'+\frac{\partial f}{\partial y'}y''+\frac{\partial f}{\partial y''}y'''+\frac{\partial f}{\partial y'''}y''''+\frac{\partial f}{\partial y''''}y'''''},$$

which, on substitution for the partial derivatives and simplification,* becomes

$$\frac{dY}{dX}=\left(y'-\frac{3y''^2}{y'''}\right)$$

* The factor which is removed is the left member of Monje's differential equation of the conic viz. $(40\,y'''^3-45\,y''\,y'''\,y''''+9\,y''^2\,y''''')$. This means merely that the curve should be of order >2.

an expression which does not involve derivatives of order greater than those involved by (A). Again differentiating the last equation with respect to X we get

$$Y'' = \frac{(3y''y'''' - 5y'''^2)}{y'''(40y'''^3 - 45y''y'''y'''' + 9y''^2y''''')}\frac{d}{dx}\left(y' - \frac{3y''^2}{y'''}\right)$$

which does not involve derivatives of order greater than or equal to six. It is clear that in general

$$\text{(B)} \qquad Y^{(n)} = \psi(x, y, y', y'', \cdots, y^{(n+3)})$$

Thus since $Y^{(n)}$ is independent of $y^{(n+4)}$ and higher derivatives, therefore (A) is a contact transformation of order four.

Now consider the equation

$$\begin{vmatrix} x^2 - x_1^2 & (xy - x_1y_1) & y^2 - y_1^2 & 2(x - x_1) & 2(y - y_1) \\ x_1 & x_1y' + y_1 & y_1y' & 1 & y' \\ 1 & x_1y'' + 2y' & y_1y'' + y'^2 & 0 & y'' \\ 0 & x_1y''' + 2y'' & y_1y''' + 3y'y'' & 0 & y''' \\ 0 & x_1y'''' + 4y''' & y_1y'''' + 4y'y''' + 3y''^2 & 0 & y'''' \end{vmatrix} = 0$$

where y', y'', y''' and y'''' denote derivatives of the curve $y = f(x)$ with respect to x calculated for the point (x, y) on it. This equation, which is obtained by eliminating the quantities a, h, b, g, f from

$$a(x^2 - x_1^2) + 2h(xy - x_1y_1) + b(y^2 - y_1^2) + 2g(x - x_1) + 2f(y - y_1) = 0$$

and the four equations resulting from this by four successive differentiations with respect to x, gives the osculating conic of the curve $y = f(x)$ at the point (x_1, y_1) on it, the derivatives y', y'', y''' and y'''' at that point being the same for the conic as for the curve. Transferring the origin to the point (x_1, y_1) the coefficients will have the values:

$$a = 3y'^2y''y'''' - 4y'^2y'''^2 - 6y'y''^2y''' + 9y''^4$$
$$h = 4y'y'''^2 + 3y''^2y''' - 3y'y''y''''$$
$$b = 3y''y'''' - 4y'''^2$$
$$g = 9y'y''^3$$
$$h = -9y''^3,$$

whence the co-ordinates of the centre are obtained as

$$\left(\frac{-3y''y'''}{3y''y'''' - 5y'''^2}, \frac{3y''(3y''^2 - y'y''')}{3y''y'''' - 5y'''^2}\right)$$

Transferring back to former origin, the co-ordinates (X_1, Y_1) of the centre of aberrancy are given by

$$\text{(C)} \quad X_1 = x_1 - \frac{3y''y'''}{3y''y'''' - 5y'''^2}; \quad Y_1 = y_1 + \frac{3y''(3y''^2 - y'y''')}{3y''y'''' - 5y'''^2}.$$

Now let c_1, c_2 be two curves having contact of order four at a point (x, y) so that the point and the derivatives y', y'', y''' and y'''' at that point are common to them. The transformation (A) will carry them over into the loci l_1, l_2 of their centres of aberrancy, as shown by (C), and these loci will have the point (X, Y), which corresponds to (x, y), in common between them since (A) involves derivatives up to order four only. Further, l_1 and l_2 should touch each other at (X, Y) since the transformation has contact property. In general (B) leads us to the

THEOREM: "*If two curves c_1, c_2 have contact of order n, $n \geq 4$, at a point, then the loci l_1, l_2 of their centres of aberrancy will have contact of order $(n-3)$ with each other at the corresponding point.*"

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GLEANINGS

This [graded catapults to throw stones and darts to various distances] was not the greatest danger. Archimedes had placed lofty and strong machines behind the walls, which suddenly letting fall vast beams, with an immense weight at the end of them, upon the ships sunk them to the bottom. Besides this, he caused an iron grapple to be let out by a chain; and having caught hold of the head of a ship with this hook, by means of a weight let down within the walls, it was lifted up and set upon its stern and held so for sometime; then by letting go the chain, either by a wheel or a pulley, it was let fall again, with its whole weight either on its head or side, and often entirely sunk. At other times the machines dragging the ship towards the shore by cordage and hooks, after having made it whirl about a great while, dashed it to pieces against the points of the rocks, which projected under the walls, and thereby destroyed all within it. Galleys frequently seized and suspended in the air, were whirled about with rapidity, exhibiting a dreadful sight to the spectators, after which they were let fall into the sea, and sunk to the bottom with their crew.

From ROLLIN: *History of the Ancients.*

THALES OF MILETUS

He was the first of the Greeks that studied astronomy. He had exactly foretold the time of the eclipse of the Sun that happened in the reign of Astyages, king of Media.

He was also the first that fixed the term and duration of the solar year among the Grecians. By comparing the bigness of the Sun's body with that of the Moon, he thought he had discovered that the body of the Moon was in solidity but the 720th part of the Sun's body and consequently that the solid body of the Sun was 700 times bigger than the solid body of the Moon. This computation is very far from the truth * * * When Thales travelled in Egypt, he discovered an easy and certain method for taking the exact height of the pyramids, by observing the time when the shadow of our body is equal in length to the height of the body itself * * * As he was one day walking, and very attentively contemplating the stars, he chanced to fall into a ditch. "Ha!" says a good old woman who was by, "How will you perceive what passes in the heavens and what is so infinitely above your head, if you cannot say what is just at your feet, and before your nose?"

From ROLLIN: *History of the Ancients.*

REVIEWS

Finite Geometrical Systems

Six public lectures by Professor F. W. Levi, Dr. Phil. Nat., Hardinge Professor of Mathematics, Calcutta University; Published by the University, 1942.

We welcome this timely monograph published by a premier Indian University, on a subject in which there is visible progress from day to day. It is a fascinating theme—combinatories—the learned professor has chosen to expound, though not without some misgivings that quick results may not be forthcoming from the mathematical investigations to satisfy the statistician's need of incomplete block designs. Briefly put, an incomplete block design is a system of b blocks containing k varieties each, selected out of v varieties so that each occurs with frequency r and each pair of varieties occurs with frequency λ among the b blocks. The mathematician in his aesthetic urge has developed such designs under various circumstances some of which are put lucidly before the reader in half a dozen talks of equal length.

The reader is first introduced to 'regular graphs' consisting of segments joining points red and blue, denoting varieties and blocks respectively. This colourful representation strongly reminds one of the early Indian efforts, notably Bhaskara's, to give the unknown quantities in algebra colour-names, such as *Kalaka*, *Nilaka*, *Lohitaka*, and *Pitaka*. From a theorem on 'even graphs', necessary and sufficient conditions for the existence of symmetric designs are deduced and illustrated by simple examples.

Again these designs are considered as general finite plane projective geometries, every geometry of this kind with its conics (quadrics), etc., furnishing a block design. Galois fields furnish the clue to the actual construction of such geometrical designs in a certain class of cases.

A step forward is taken to review the possible contributions of combinatorial topology. If one considers the points as varieties and the cells as blocks, every regularly subdivided topological surface of α_0 points, α_1 segments and α_2 cells can be considered as a block system, with $v=\alpha_0$, $b=\alpha_2$, $r=a_0$, each point being incident with a_0 segments and each cell with $a_2=k$ segments subject to the conditions

$$a_0\alpha_0=2\alpha_1=a_2\alpha_2,\ \lambda(v-1)=r(k-1),\ K=2-\alpha_0+\alpha_1-\alpha_2>0, \text{ and } \lambda\geq 2$$

Among the block designs deduced from such surfaces we are glad to note one which has been missed in the tables of Fisher and Yates, viz.,

127, 347, 567, 432, 614, (135), 236
237, 457, 167, 125, 341, (246), 563

corresponding to $v=7$, $k=3$, $b=14$, $r=6$, $\lambda=2$, of which the blocks in

brackets are missing in the text and supplied by the reviewer. It is interesting to note that this design is made up of two similar designs as arranged above in separate rows, the design represented in the second row being simply a duplicate of that in the first row under a different permutation of the digits. Professor Levi submits that the design ($v = 9$, $b = 24$, $k = 3$) corresponding to $a_0 = 8$ is not known to him. However, it may be pointed out that this unknown case has been dealt with partially by R. C Bose * who says that it can be obtained by duplicating his design $v=9$, $b=12$, $k=3$. But here a warning may be necessary. Mere duplication is rather crude. It will be better to think of a proper substitution operating on the symbols so as to get a new set of blocks having no block in common with the original set. It is possible that such a substitution does not exist.

Sometimes what is topologically impossible gives a satisfactory block design while the principle of topological duality between points and cells is inadmissible in an unsymmetrical block design, for if $v \neq b$ the interchange of v and b on the one hand and r and k on the other will violate the λ-condition. Prof. Levi is right in conceiving the λ-condition as a real stumbling block to the use of topological armoury by the statistician. Graphs are perhaps more serviceable.

The statistician must be grateful to the warning that too much faith cannot be put in topological proofs which may contain an undetected fallacy. For example, the theorem regarding the possibility of Euler squares of order $4n-2$ (n being a positive integer) has no satisfactory proof in Prof. Levi's opinion. But statisticians like F. Yates† seem to believe in the theorem.

The fourth lecture discusses the existence of geometrical designs which have no analogues in two dimensional projective geometry, such as non-Desarguesian geometrical designs, where the straight lines do not possess the familiar Euclidean straightness but may be bent anyhow on account of the relaxing of some condition like the commutative law of multiplication. It is shown that this commutative law of multiplication is a luxury for Desargues's theorem. For non-Desarguesian geometries one can relax the conditions still further by removing the associative law of multiplication and one of the distributive laws also. In this way by progressive grants of enfranchisement almost new species of geometries are ushered into existence, involving, of course, concomitant complexities in mathematics. Suddenly, however, some unforeseen simplifications arise, such as Wedderburn's theorem that every finite skew field is a field ; and Desargues' theorem holds in every geometry of higher dimensions than two. The next two lectures take us still further afield to systems of elements based on Abelian groups, non-isomorphic to Galois fields and doubly transitive groups of degree p^n and order $p^n(p^n-1)$ in which the subgroup not displacing any particular object is non-commutative.

* R. C. Bose (1939) *Annals of Eugenics.*, 9, pt 4, 373, 377.

† F. Yates., Imperial Bureau of Soil Science, Technical Communication No. 35. *The Design and Analysis of Factorial Experiments*, page, 82.

The sixth and the last lecture introduces the reader to alternating fields and rings in which the commutative and associative laws of multiplication are both relaxed while the two distributive laws are held intact. A mild form of the associative law still prevails as $a(ab)=(aa)b$. Miss Ruth Moufang is ruthless in her scrutiny and investigation of the different aspects of Desargues's theorem. Prof. Levi quotes at some length from her researches and the text here becomes a little obscure to the general reader, who is apt to lose patience but for the comforting prospect that the book closes in a couple of pages more. The last announcement in the text is the theorem that a finite alternating ring containing no divisors of zero is a Galois field. The final moral is that there are four types of geometries:

(i) Geometry over any field in which all the incidence theorems of ordinary geometry hold; (ii) geometry over a skew field where Desargues's theorem holds but not Pappus's theorem; (iii) semi-Desarguesian geometry in which harmonic points exist and the theorem of the complete quadrilateral holds, corresponding to an alternating but not skew field; and (iv) geometry in which harmonic points coincide and Desargues's theorem completely fails.

All this feast of geometries is bound to inspire the statistician with new *vigour to carry on his own 'combinatories', partaking of such assistance as he* may get from the pure mathematician who is interested more in showing where the quarry lies than in actually digging out particular results

A few errors of omission in print as, for example, in pages 17 and 48, some alien turns of expression like polyga and conics osculating a straight line, the absence of a select bibliography—these are little defects which can be easily set right in a subsequent edition.

Mysore, A. A KRISHNASWAMI AYYANGAR.

ARCHIMEDES TOMB

Archimedes, by his will, had desired his relations and friends to put no other epitaph on his tomb, after his death, than a cylinder circumscribed by a sphere; and to set down at the bottom the proportion which those two solids, the containing and the contained, have to each other. He might have filled up the bases of the columns of his tomb with relievos, whereon the whole history of the siege of Syracuse might have been carved and himself appeared like another Jupiter thundering upon the Romans. But he set an infinitely higher value upon a discovery, a geometrical demonstration, than upon all the so-much celebrated machines which he had invented.

Hence he chose rather to do himself honour in the eyes of posterity, by the discovery he had made of the relation of a sphere to a cylinder of the same base and height; which is as two is to three.

From ROLLIN: *History of the Ancients.*

Galois Lectures

BY JESSE DOUGLASS, PHILLIP FRANKLIN, C. J. KEYSER AND LEOPOLD INFELD, Scripta Mathematica Library No. 5, New York 1941, 124 pp; Price $ 1. 25.

The mathematical public owes to the organisers of the *Scripta Mathematica*, not only an attractive quarterly, but also a series of readable booklets intended to give a birds-eye-view of some of the most interesting regions within and on the border of the subject which no mathematical tourist should miss. The booklet under review, is the fifth of the series and contains 4 lectures delivered at the Galois Institute of Mathematics, Long Island University, Brooklyn, N. Y., whence the title of the book.

In the first address Prof. Jesse Douglas gives a Survey of the Theory of integration considering therein the integrals associated with the names of Riemann, Stieltjes, Lebesgue and Denjoy, the last receiving only a brief mention. Criteria for existence and the properties of the integrals are given with simple examples, and improper and multiple integrals also come in for consideration with a reference to their connection with Fourier Series and Transforms. References are given to standard books so that after the "survey", the reader may dwell "happily ever after" in one of these regions.

The next lecture by Prof. Phillip Franklin deals with the intriguing "Four Colour Problem" enunciated in 1850 by De Morgan e.g. that every geographical map on a plane or a sphere can be coloured in 4 colours, so that no two regions which have a boundary in common are coloured alike. Experiments with diverse maps support this result but there is no mathematical "proof" yet. Heawood proved in 1890 that 5 colours suffice, but the reduction, from 5 to 4 is one of extra-ordinary difficulty and has been nibbled at by a large number of mathematicians, and painfully extended till in 1940 Winn proved that for all maps of 35 or fewer regions 4 colours suffice. In other cases, the possibility of colouring with 4 colours can be proved at present only if the map contains certain special configurations such as a region with 4 sides, rings of 4 or fewer regions, a ring of 5 regions not surrounding a single pentagon etc. For being capable of colouration in 2 colours only, like a chess board a necessary and sufficient condition is that through each vertex an even number of edges pass, while for 3 colours the corresponding condition is that each region has an even number of sides. An example of the latter is a set of hexagons arranged like the cells in a beehive.

When we pass on to multiply connected surfaces like the anchor ring, the minimum number of colours is 7, while for a one sided surface like the projective plane the "chromatic number" is 6. In higher dimensions, the problem becomes trivial since even in 3-space there is no limit to the number of colours required. To see this take 2 layers of long square prisms the top layer at right angles to the bottom layer fastening the n^{th} prism of the top layer with the n^{th} prism of the bottom layer and regarding each such pair as a single block. Each block then touches every other and should be coloured differently from the others.

Problems equivalent to the 4 colour problem are also mentioned. For example "The colouring of a regular map in 4 colours is equivalent to the colouring of its edges in 3 colours."

In the third lecture Prof. C. J. Keyser points out how Charles Sanders Pierce, (1839—1914) the son of Benjamin Pierce to whom we owe the definition of mathematics as "the Science which draws necessary conclusions", was a pioneer to whom the world owes many basic seed-thoughts which were later to become fruit giving trees. These seed thoughts are those connected with "pragmatism", "propositional function", the "logic of relations" and the infinities of cantor.

In the last lecture on "The Fourth Dimension and Relativity" Prof. Infeld gives in the form of a conversation the concept of the 4-dimensional space-time continuum as a static picture of the world of events.

A. N. RAO

Projective Geometry

By S. Suryanarayana Iyer, M.A. and A. Santiago, M.A. with a foreword by Rev. C. Racine, S.J., St. Joseph's Industrial School Press. Trichinopoly, 1942, pp. 374.

This is a laudable attempt on the part of the authors to produce a text-book suitable for the B. A. Pass and Honours classes of the South Indian Universities, covering roughly the same ground as Askwith's *Pure Geometry.* Some of the topics receive ampler treatment such as Newton's Theorem and plane perspective. A fallacy in the direct application of Newton's Theorem with one of the directions parallel to an asymptote was pointed out by the first of the two authors in *The Mathematics Student* vol. IV p. 82.

One welcome improvement over the usual treatment is the illustrative "Analytical treatment" of the results in pure geometry, though this might with advantage have been carried still further, for example in the treatment of pencils and ranges of conics. The appropriate coordinate system for projective geometry is homogeneous coordinates, whether for points on a line or on a plane, and there is no need for the restriction $x+y+z=1$ given on page 322. Several of the results given in the chapter on circular points at infinity which are very properly treated by homogeneous co-ordinates, will be true only if the coefficients of the conic are assumed to be real. The authors devote one chapter to "involution" and long afterwards, another to "homographic ranges and pencils" apparently bearing no relation to the former, separate analytical treatments being given in the two cases. The homographic transformation or the linear operator which is the connecting link receives but scant recognition, if any at all.

The Syllabus in Geometry followed in many Indian Universities is in urgent need of revision. While the characteristic properties of conics may be allowed to be derived from those of a circle by projection, enough material should be included to show that Projective Geometry, far from being the handmaid of Metrical Geometry is a Queen in her own fair and extensive

domain which includes the metrical as a province thereof. Such a treatment would use analytical methods very freely with homogeneous co-ordinates, lay proper emphasis on the linear transformation, projective co-ordinates and the group concept, and if written so as to be suitable for use as a text book, should prove a stimulus for a revision of our courses. In remarking that the book under review is not of the above type, we are only stating that the authors have shown worldly wisdom and economic insight by meeting an existing need instead of seeking to create one. Still, one wishes there were less of worldly wisdom in the world!

A. N. RAO

A Manual of Geometry

By N. K. Narasimhamurthy M. Sc., pp. 120 Price Rs. 1—4—0.
The Prabhakar Book Depot, Bangalore.

A Manual of Trigonometry

By N. K. Narasimhamurthy M. Sc., pp. 176+16. Price Rs. 2—0—0.
The Prabhakar Book Depot, Bangalore.

Both these books are intended for the students of the Intermediate classes of the Mysore University. The Geometry Manual deals with the geometrical representation of algebraic identities, medial section, ratio and proportion, similar figures and some propositions in elementary solid geometry. The Trigonometry manual covers much the same ground as Loney's Trigonometry, Part I and contains 16 pages of mathematical tables. The treatment in both the books is brief but clear, and there are explanatory notes to bring out points requiring special attention. The printing and get up are good, and the books should be useful to those for whom it is intended.

A. N. RAO

The Syracusans, who had been in former times so fond of the sciences, did not long retain the esteem and gratitude they owed to a man who had done so much honour to their city. Less than a hundred and forty years after, Archimedes was so perfectly forgotten by his citizens, notwithstanding the great services he had done them, that they denied his having been buried at syracuse. It is Cicero who informs us of this circumstance.

At the time that he was quaestor in sicily, his curiosity induced him to make a search after the tomb of Archimedes. The syracusans assured him that his search would be to no purpose, and that there was no such monument among them. Cicero pitied their ignorance *** At length, after several fruitless attempts he pereived without the gate of the city, *** a pillar almost entirely covered with thorns and brambles, through which he could discern the figure of a sphere and a cylinder.

FROM ROLLIN: *History of the Ancients*.

ANNOUNCEMENTS AND NEWS

The following gentlemen have been elected members of the Indian Mathematical Society:

M. Abdulla Butt, Esq., M. A., Lecturer, Muslim University, Aligarh.

C. P. S. Menon, Esq., M.A., M.Sc., F.R.A.S., *the Doon School, Dehra Dun.*

Ram Dhar Misra, Esq., M.Sc, Ph.D. (Edin) Lecturer, Lucknow University.

Akhowry Vindhyachal Prasad, Esq., M.Sc., Lecturer, Rajendra College, Chapra (Bihar).

P. Thirumalachary, Esq., B.Sc. (Hons.) C. M. A's Office, Poona.

V. V. Kale, Esq., B.E., Asst. Engineer, P. W. D., Bombay.

A. S. Ranganathan, Esq., B.A., 12 A, Cleaveland Town, F Street, Bangalore Cantonment.

R. K. Rubugunday. Esq., B.A. (Cantab), ¼ North Mada Street, Mylapore. Madras.

Ambikeshwar Sharma, Esq., Birla College, Pilani (Jaipur State).

C. B. Rathie, Esq., Asst. Prof. of Mathematics, Dungar College Bikaner (Rajaputana).

F. J. Noronha B.A., Hons. (Lond), Lecturer, Central College, Bangalore.

K. N. P. Nair, Esq., Doon College, Dehra Dun.

V. Ramaswami B.A. Hons. (Madras), Ph.D. (Cantab), Reader, Andhra University, Guntur.

The Managing Committee of the Indian Mathematical Society has been reconstituted as follows:

President: Dr. F. W. Levi, Hardinge Professor of Mathematics, Calcutta University.

Secretaries: Prof. S. Mahadevan, Presidency College, Madras.
Dr. M. R. Siddiqi, Osmania University, Hyderabad.

Librarian: Prof. R. P. Shintre, Fergusson College, Poona.

Treasurer: Prof. L. N. Subramaniam, Christian College, Tambaram.

Other Members: Prof. D. D. Kosambi, Fergusson College, Poona.
" J. Maclean, Wilson College, Bombay.
" A. N. Singh, Lucknow University, Lucknow.
" A. Narasinga Rao, Annamalai University, Annamalainagar.
" Ram Behari, St. Stephen's College, Delhi.
" N. R. Sen, Calcutta University, Calcutta.
" S. S. Pillai, Calcutta University, Calcutta.

The authorities of the Annamalai University have invited the Indian Mathematical Society to hold its next conference at Annamalainagar about the end of December 1943, and the Society has accepted the invitation. It is also proposed to hold, in connection with the conference, a mathematical exhibition intended to illustrate the richness and variety of the subject and the wide range of its applicability to life situations. Suggestions regarding suitable items, as well as charts, models, instruments and other exhibits will be gladly received and exhibits on loan duly acknowledged and returned at the end of the conference. It is also proposed to have a "book section" for exhibiting books on mathematics. All correspondence relating to the conference, and all papers to be read at the session may kindly be sent (with 2 short abstracts of each paper) to Dr. A. Narasinga Rao, Annamalainagar P. O, South India. It is hoped that, in spite of the difficult conditions under which the conference and exhibition are held the enthusiasm and cooperation of the members will make the venture a great success.

The Society has decided to suspend the Life Composition concessions till the end of the war.

The Ramanujam Memorial Prize of the Madras University for 1941 has *been awarded jointly to Dr. S. Minakshisundaram and Dr. T. Venkatarayudu* the titles of the winning theses being repsectively "A problem in Fourier expansion" and "Vibrations of a symmetric point system."

The University of London has conferred the Degree of Ph. D. on S. M. Shah, Esq., Lecturer in Mathematics in the Muslim University, Aligarh. The title of the Thesis was "On the relations between the number of zeros and the maximum modulus of an Integral function"

Mr. Nizamuddin M. Sc. has been appointed Lecturer in Mathematics in the Muslim University, Aligarh.

Prof. A. R. Forsyth whose weighty treatises on Function Theory and on Differential Equations have played an important part in the education of generations of mathematicians died June 1942 at the age of 84. Other Mathematicians who died recently and with whose names the students of higher m athematics will be familiar include:

Dr. Vito Volterra of Rome in October 1940 aged 80 years

Prof. Alfred Pringsheim of Munich in 1941 at the age of 91 years

Dean W. C. Graustein of Harvard in January 1941

Dr. E. L. Ince, of Edinburgh in 1941 at the age of 49 years

Prof. I. Schur of Berlin died in 1941

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INTUITIONISTIC THEORY OF LINEAR ORDER

BY

K. CHANDRASEKHARAN, *Madras University*

Introduction

The two principal features of Intuitionistic Mathematics are the rejection of the unlimited validity of the principle of the excluded middle, and the requirement that mathematical proofs should be constructive and not merely based on *reductio ad absurdum.* Since most of the methods of proof in classical mathematics necessitate dealing with alternatives and therefore involve, in some sense, the principle of the excluded middle, the question may be raised as regards the method that intuitionists adopt in such instances. For the intuitionists, the unending sequence of positive integers is fundamental as it is revealed in the primordial time-consciousness; and the principle of generation of these integers is essentially of the same nature as the principle of mathematical induction. Thus, sets which are effectively enumerable occupy a special place in Intuitionism, and when we speak of 'all' the elements of such sets, the word 'all' has a definite content, which it fails to carry when applied to infinite totalities of some indefinite nature. Hence the methods of proof employed in the case of enumerable sets cannot be transferred, as they are, to the case of infinite sets in general.

The proofs that intuitionists employ are generally of two sorts (1) If a property requires to be proved, and there are only a finite number of possibilities, then the intuitionist will accept a proof as constructive if and only if a systematic proof is constructed for everyone of the possibilities, even though we may not give a systematic process to specify which of those possibilities exactly takes place. (2) In case, the number of possibilities is *enumerably* infinite, then the property is said to be proved, if it is proved for the possibility 1, and if it can be proved for the possibility $r+1$ whenever it is proved for r. The possibilities being enumerable, an exhaustion of these possibilities step-by-step is equivalent to proving the result in 'all' possibilities. Proofs of this kind are intuitionistically valid since the principle of induction used therein is a basic

intuitionistic idea, even though the demand of specifiability associated with the strictest meaning of 'or' is not fully complied with.[4]

In a recent paper [8] contributed to *The Mathematics Student*, I gave an outline of intuitionistic set-theory which typically exhibits those principal features that I have just now mentioned. The present paper is devoted to an analysis of Brouwer's theory of virtual order in its formal aspects, and contains some theorems concerning the nature of the *full product* of a *fundamental series* of virtually ordered sets, which are slightly more general than those of Brouwer [1] for the full product of sets of integers. The methods of proof employed are of the two kinds above described.

§ 1. Virtual Order

1·1. *A* set P is *virtually ordered*, if for the element pairs (a, b) of P we can define an ordering relation, denoted by $a<b$, such that the following postulates are satisfied.

P_1. $r=s$, $r<s$, $r>s$ are mutually exclusive.

P_2. If $r=u$, $s=v$, and $r<s$, then $u<v$.

P_3. If $r>s$ is impossible, and $r<s$ is impossible, then $r=s$.

P_4. If $r>s$ is impossible, and $r=s$ is impossible, then $r<s$,

P_5. If $r>s$ and $s>t$, then $r>t$.

From P_1, P_3, P_4, we observe that the relations $=$, $>$, $<$ are normal, that is, identical with their double negations. For, from P_1 we see that $r=s$ implies $r \not< s$ and $r \not> s$; while from P_3, we see that $r \not> s$ and $r \not< s$ imply $r=s$; so that, $r=s$ is the same as '$r \not< s$ and $r \not> s$', which, being the product of two negative propositions, is normal. Similarly, $>$, $<$ are also normal.

In fact, we can replace P_3 by the statement

P_3': $=$ is a normal relation,

and deduce P_3 as a consequence, from the other postulates and P_3'. For, if $=$ is normal, then it is identical with the absurdity of its absurdity; and by P_4, $r \not> s \cdot r \not< s$ imply the absurdity of the absurdity of $r=s$; that is, they imply $r=s$. So P_3 is deduced.

Similarly, P_4 can be replaced by the statement,

P_4': $<$ is a normal relation,

and P_4 can be deduced as a consequence, from P_1, P_2, P_3 and P_4'.

It should, however, be noted that we cannot simultaneously replace P_3 and P_4 by P_3' and P_4' respectively. As an example, consider the virtually ordered set M, consisting of elements m_r. Consider the set N of pairs of elements m_r. Define an ordering relation in N thus:

$$(m_r, m_s) > (m_t, m_u)$$

if and only if $m_r > m_t$ and $m_s > m_u$.

Then, the relation $>$ in N, satisfies P_1, P_2, P_5. Also, it is normal since it is the product of two normal relations $m_r > m_t$ and $m_s > m_u$. Equality is similarly normal. Thus P_3' and P_4' are satisfied. But it does not satisfy P_4 or P_3, as in the case $m_r > m_t$, $m_s = m_u$.

1·2. Brouwer generally derives a *virtual order* in a set from some previously defined, asymmetric, transitive relation satisfying certain requirements. A formal view of his derivation is as follows:

Let the elements of a set M be the field of a binary relation $\circ>$ satisfying

L_1. $a \circ> b$, $a = b$, $a <\circ b$ are mutually exclusive.

L_2. If $a = b$ and $c = d$, then $a <\circ c$ implies $b <\circ d$.

L_3. If $a \circ> b$ is absurd, and $a <\circ b$ is absurd, then $a = b$.

L_4. If $a \circ> b$, and $b \circ> c$, then $a \circ> c$.

Denote the negation of $\circ>$ by $\leqslant\circ$, and its negation (i.e. negation of $\leqslant\circ$) by $>$. Then, we may shew, as follows, that the relation $>$ satisfies the conditions P_1 to P_5.

P_1: From L_1 we observe that $a \circ> b$, $a = b$, $a <\circ b$ are mutually exclusive, and so their double negations are also mutually exclusive. The double negation of $\circ>$ is $>$, and the double negation of $=$ is itself, since equality is normal by L_3. Hence $a > b$, $a = b$, $a < b$ are mutually exclusive, which is P_1.

P_2: If $a = b$, and $c = d$, then $a < c$ implies $b < d$. For, by L_2, we see that $a <\circ c$ implies $b <\circ d$, and therefore the double negation of $a <\circ c$ implies the double negation of $b <\circ d$. That is, $a < c$ implies $b < d$.

P_3: $a \geqslant b$ and $a \leqslant b$ imply $a = b$. This follows from L_3, if we note that $\geqslant$ is the same as $\circ\geqslant$.

P_4: $a \leqslant b$ and $a \neq b$ imply $a<b$. Since $a \leqslant b$ is the same as $a \leqslant^{\circ} b$, suppose that $a \leqslant^{\circ} b$ and $a \neq b$. Then, if $a \circ\!\geqslant b$, by L_3 we have $a=b$, which contradicts $a \neq b$. Hence $a \circ\!\geqslant b$ is impossible. So $a < b$.

P_5: If $a > b$, and $b > c$, then $a > c$. To prove this, we assume $a > b$, $b > c$, and prove that $a \neq c$ and $a \circ\!\geqslant c$, which together imply $a > c$, by P_4. Firstly, $a \neq c$. For, if $a=c$, then, since $b > c$, we would have $b > a$ by P_2, which contradicts the hypothesis. Secondly, $a \circ\!\geqslant c$. For, if $a <^{\circ} c$, then $b \circ\!\geqslant a$, for, otherwise $b <^{\circ} c$ which contradicts $b > c$. On the other hand, since $a > b$, we have $a \circ\!\geqslant b$. Thus the assumption $a <^{\circ} c$ leads to $b \circ\!\geqslant a$ and $a \circ\!\geqslant b$; that is, $a=b$, which contradicts the hypothesis $a > b$. Thus $a <^{\circ} c$ is impossible. So $a \circ\!\geqslant c$, which is the same as $a \geqslant c$.

N.B.—Since the virtual order '$>$' is obtained only as the double negation of the given asymmetric transitive relation $\circ\!>$, it follows that, even if we had taken another relation $\circ\circ\!>$ which implies $>$ and is implied by $\circ\!>$, and which satisfies the postulates L_1 to L_4, then it follows from a known property of normal elements that the double negation of that relation also will define the same virtual order.

1·3. (1) A *virtually ordered* set is said to be *ordered*, if we have a systematic process for determining which of two *different* elements r, s is the greater.

A set is said to be *discrete* if we can determine by a systematic process whether any two given elements are equal or not.

A *discrete ordered* set is said to be *completely ordered.* This notion is introduced in order to bring out the speciality of the set of integers among virtually ordered sets in general. The principle of generation of the integers contains a general method for determining whether two integers are equal, and if not, which of them is the greater.

We can also see that a *virtually ordered* set is not necessarily *ordered.* Suppose k_1 is the rank of that digit in the decimal expansion of π at which, for the first time, the sequence 0123456789 commences. Suppose $a > b$ if k_1 exists, or it is impossible that k_1 does not exist, and $a < b$ if it is impossible for k_1 to exist. Now the set consisting of the two different elements a, b is virtually ordered. But it is not ordered since we cannot decide which of them is the greater one.

If a, b are two elements of a virtually ordered set P, then the *closed interval* $[a, b]$ means the set of elements c of P for which neither $c < a$, $c < b$ nor $c > a$, $c > b$, can hold. a and b are called the end elements of $[a, b]$.

By the *open interval* (a, b) is meant the set of elements c of P which lie between a and b; that is, firstly they are different both from a and from b, and secondly they belong to the closed interval $[a, b]$. If $a < b$, then $a < c < b$.

The virtually ordered set P is *everywhere dense* if, between any two different elements of P there exist elements of P, and *everywhere dense in a restricted sense* if, in addition, an element of the set can be constructed and, to the left and right of any element of P, there exist other elements.

This latter notion is introduced to bring out the speciality of rational numbers among everywhere-dense sets in general. This can be seen easily by considering the rational numbers as pairs of mutually prime integers. Since the integers are themselves *completely ordered*, we can decide whether two pairs are identical or different, and hence the set is *discrete*. It is also *ordered* for, given two *different* elements $\frac{p}{q}, \frac{r}{s}$, we can verify whether $ps - rq$ is positive or negative, and say that $\frac{p}{q}$ is the greater if $ps - rq$ is positive. Thus, the set of rational numbers is *completely ordered*. It is *everywhere dense* since the middle point, for instance, of any interval with rational end elements, is a rational number. It is dense in the restricted sense, since we can produce a rational number from a pair of relatively prime integers, and obtain other elements which are greater, or less than the given one.

If between two virtually orderd sets P and Q, a biunivocal correspondence is established which leaves the order invariant, then we say that P and Q possess the same *order-type*, or they are *similar*.

The set of positive integers in their natural order is said to possess the order-type ω, and ordered sets of order type ω are called *fundamental series*.

In a virtually ordered set M, we have a *divergent net of closed intervals* $i_1, i_2, \ldots$ where $i_{\nu+1} \subset i_\nu$ if to each p there is a ν_p such that p cannot belong to i_{ν_p}; and a *convergent net with kernel* q, if every two

intervals of the series are different, q belongs to every interval, and any element belonging to every interval is identical with q.

An element which is the *kernel* of a convergent net is called *principal*. If all elements are principal, then M is *dense-in-itself*.

If in M it is impossible for a divergent net to exist, then M is *closed*. M is *perfect*, if it is *closed* and *dense-in-itself*.

1·4. Examples:

The set of positive integers is completely ordered and nowhere dense, while the set of rational numbers is also oompletely ordered but everywhere dense. The set of positive integers is closed, but not dense-in-itself.

The set of positive integers (k_n) which are the ranks of the digits *at which the sequence 0123456789 commences in the decimal expansion* of π, is completely ordered.

We can define a set which is everywhere dense, but not dense in a restricted sense.

Let M be the set of rational numbers greater than an element of the set (k_n). Then M is everywhere dense, but it is not dense in a restricted sense, since we cannot construct any element of the set for the simple reason that we do not know whether the set (k_n) is null or not.

§ 2. Full Product of Virtually ordered Sets

Consider a finite number or a fundamental series of virtually ordered sets $M_1, M_2, M_3, \cdots$ which are non-null. Let the order types of these be $m_1, m_2, m_3, \cdots$ Denote an element of M_ν by p_ν and consider the set $M \equiv V(\cdots\cdots, M_3, M_2. M_1)$ of complexes $f \equiv (p_1, p_2, \cdots)$. The set M is called *the full product of factors* $\cdots$ $\cdots M_3, M_2, M_1$, when it is virtually ordered as follows:

$f' \circ\!> f''$ if there exists an r such that $p_r' > p_r''$ and $p_\nu' \geqslant p_\nu''$ for $\nu < r$. Then, the relation $\circ\!>$ satisfies L_1, L_2, L_3, L_4. Denote the double negation of $\circ\!>$ by $>$. Then the new relation '$>$' defines a virtual order, and $\geqslant$ defined as negation of $<$, is the same as $\circ\!\geqslant$.

The order type of M depends only on the order types of the M_ν's and is denoted by $V(\cdots\cdots m_3, m_2, m_1)$ and called the *full product* of factors $\cdots m_3, m_2, m_1$.

The full product is *associative*; that is, the sequence, ... M_3, M_2, M_1, can be divided into groups $N_r = (M_{\nu_{r-1}+1}, \ldots M_{\nu_r})$, where $\nu_1 = 1$, $\nu_2, \nu_3, \ldots$ is a finite or infinite sequence of increasing positive integers; every N_r and every product of N_r's like $V(N_r, N_{r-1}, \ldots N_1)$ can be virtually ordered according to the order-laws in M_ν. On the other hand, the original full-product M can be considered as the full product of M_ν's or as the full product of N_r's, and whichever point of view is adopted, the same virtual order results; and this is what is meant by *associativity*.

Let any element of M be expressed as $(z_1, z_2, \ldots)$ where z_ν, for any ν, is an element of the virtually ordered set N_ν. This leads to a virtual order in M, the relations of which can be denoted by $\dot{=}$, $\circ\dot{>}$, $\dot{>}$, while the relations $=$, $\circ>$, $>$ pertain to the initial virtual order.

The equivalence of $=$ and $\dot{=}$ is evident. We shall prove that $\leqslant$ and $\dot{\leqslant}$ are also equivalent.

Firstly we prove that $f' \leqslant f''$ implies $f' \dot{\leqslant} f''$. For that, suppose $f' \leqslant f''$. Then, for any r, $(z_1', z_2', \ldots z_r') \leqslant (z_1'', \ldots z_r'')$, for, $(z_1', \ldots z_r') \circ> (z_1'', \ldots z_r'')$ would imply $f' \circ> f''$ which contradicts $f' \leqslant f''$. Next assume that, if possible, $z_1' \geqslant z_1''$, $z'_{r-1} \geqslant z''_{r-1}$ $z_r' > z_r''$ hold simultaneously. Then, we have,

(i) $(z_1', \ldots z_r') <\circ (z_1'', \ldots z_r'')$ is impossible, for if it were true, then $z_\nu' <\circ z_\nu''$ for at least one $\nu \leqslant r$ which would contradict the assumption made.

(ii) $(z_1', \ldots z_r') = (z_1'', \ldots z_r'')$ is impossible, since $z_r' > z_r''$.

Thus, we have, $(z_1', \ldots z_r') > (z_1'', \ldots z_r'')$ since $\leqslant\circ$ is the same as $\leqslant$, and this contradicts the hypothesis.

Hence our assumption that

$$z_1' \geqslant z_1'', \; z_2' \geqslant z_2'', \qquad z_{r-1}' \geqslant z''_{r-1}, \; z_r' > z_r''$$

hold simultaneously for any r is absurd. That is, $f' \dot{>} f''$ is absurd. Therefore $f' \dot{\leqslant} f''$.

Next, we prove that $f' \dot{\leqslant} f$ implies $f' \leqslant f''$. By the above reasoning, $f' < f''$ implies $f' \dot{\leqslant} f''$, and we know that $f' < f''$ implies

$f' \neq f''$. Thus, $f' < f''$ implies that $f' \dot{<} f''$. In other words, $f' \dot{\geqslant} f''$ implies $f' \geqslant f''$. By an interchange of f' and f'', we see that, $f' \dot{\leqslant} f''$ implies $f' \leqslant f''$.

Hence $\geqslant$ and $\dot{\geqslant}$ are equivalent. It follows, therefore, that the full product is associative.

N. B. We can define $f' \circ\circ\!> f''$ if $p_r' > p_r''$ and $p_{\nu'} = p_{\nu''}$ for $\nu < r$. This is obviously an asymmetric transitive relation, satisfying $L_1 - L_4$. Also $f' \circ\circ\!> f''$ implies $f' \circ\!> f''$, and the negations of both are the same. This exemplifies the remark at the end of 1·2.

§ 3. Theorems on the Full-Product

THEOREM I. *The full product of a fundamental series of virtually ordered sets is everywhere dense provided we are certain that any M_r possesses one of the two properties viz., (A) to every element of M_r we can construct a greater element, or (B) to every element of M_r we can construct a smaller element.*

Proof: Let f', f'' denote two elements of N, where $f' = (m_1', m_2', \cdot \cdot)$; $f'' = (m_1'', m_2'', \cdot \cdot \cdot \cdot)$. Assume that f' is different from f''. That is, it is impossible that $m_r' = m_r''$ for every r. So we can say that $m_1' = m_1''$, $m_2' = m_2''$, $\cdot \cdot \cdot$ $m_k' = m_k''$, and $m'_{k+1} > m''_{k+1}$ or $m'_{k+1} < m''_{k+1}$ hold simultaneously for *some* value of $k = 0$, or 1 or $2 \cdot \cdot \cdot$, since these exhaust all the possibilities step by step, though we may not be able to *specify* the value. Let it be true for $k = \nu - 1$, where ν is not previously given, but may take any positive integral value. For every such possible value of ν, we shall show a regular method of constructing $f''' \equiv (m_1''', m_2''', \cdot \cdot \cdot \cdot)$ so as to lie between f' and f''.

The components of f''' are constructed by the following specification:

$$m_s''' = m_s' = m_s'' \text{ for } s \leqslant \nu - 1.$$

If $m_\nu' < m_\nu''$, and ν is such that $M_{\nu+1}$ possesses the property (A), then $m_\nu''' = m_\nu'$, and $m'''_{\nu+1}$ is an element of $M_{\nu+1}$ constructed such that it is $> m'_{\nu+1}$.

If $m_\nu' < m_\nu''$, and ν is such that $M_{\nu+1}$ possesses the property (B), then $m_\nu''' = m_\nu''$, and $m'''_{\nu+1}$ is an element constructed so as to be $< m_{\nu+1}''$.

If $m_\nu'' < m_\nu'$, and ν is such that $M_{\nu+1}$ possesses the property (A), then $m_\nu''' = m_\nu''$, $m_{\nu+1}'''$ is an element constructed so as to be $< m_{\nu+1}''$.

If $m_\nu'' < m_\nu'$, and ν is such that $M_{\nu+1}$ possesses the property (B), then $m_\nu''' = m_\nu'$, $m_{\nu+1}'''$ is an element constructed so as to be $> m_{\nu+1}'$.

The other places in f''' can be filled by any components.

N.B.—The proof of the theorem does not require the specifiability of ν since all the possibilities are provided for, and these possibilities can be enumerated, and there is a regular method of constructing f''' in every possible situation. Cf. *Introduction.*

THEOREM II: *The full product of a finite number or a fundamental series of dense factors is dense.*

Proof: Let $N = V(\cdots\cdots, M_2, M_1)$, where N consists of elements $n = (m_1, m_2, \cdots\cdots)$, where m_r is an element of M_r.

Given two elements of the set N which are different, we should give a method of constructing another element of the set between them. Let the given elements be

$$n' = (m'_1, m'_2, \cdots\cdots) \text{ and } n'' = (m_1'', m_2'', \cdots\cdots).$$

Since $n' \neq n''$, we know that $m_\nu' = m_\nu''$ *not* for all ν. Therefore, there is an r (unspecifiable, as in the argument of TH. I). Such that $m_\mu' = m_\mu''$ for $\mu \leqslant r$ and $m_{r+1}' > m_{r+1}''$ or $m_{r+1}' < m_{r+1}''$, where $r = 0$ or 1 or $2 \cdots\cdots$ Whether $m_{r+1}' > m_{r+1}''$ or $m_{r+1}' < m_{r+1}''$ we can construct an element to lie between m_{r+1}' and m_{r+1}'', since *each* M_r is dense. Call that element m_{r+1}''', so that either $m_{r+1}' < m_{r+1}''' < m_{r+1}''$, or $m_{r+1}'' < m_{r+1}''' < m_{r+1}'$. In either case construct $n''' = (m_1', m_2', \cdots m_r' \; m_{r+1}''', \cdots\cdots)$. It is clear that n''' lies between n' and n''.

THEOREM III: *The full product N is dense in itself, provided we are certain that all the factors M_ν possess either property (A) or property (B) of Th I.*

Proof: Let any element of N be denoted by $n = (m_1, m_2, \cdots\cdots)$ If M_ν has the property (A), construct $n'_\nu = (m_1, m_2, \cdots\cdots)$ and $n''_\nu = (m_1, m_2, \cdots m'_\nu, \cdots)$, where $m'_\nu > m_\nu$.

If M_ν has the property (B), construct $n'_\nu = (m_1, m_2, \cdots m''_\nu \cdots)$, where $m''_\nu < m_\nu$, and $n''_\nu = (m_1, m_2, \cdots\cdots)$.

Denote the closed interval $[n'_\nu, n''_\nu]$ by i_ν. It is clear that n is contained in every i_ν. On the other hand, if any element $k = (k_1, k_2, \cdots)$ belongs to every i_ν then we can shew that $k = n$. To prove this, it is sufficient to shew that $k_\nu = n_\nu$ for each ν. Now, if k belongs to any i_ν, then its first $\nu - 1$ components are respectively equal to the first $\nu - 1$ components of n. Since k is contained in every i_ν, every one of its components is equal to the corresponding component of n. That is, $k = n$.

THEOREM IV: *The full product is dense in itself if one of its factors is dense in itself.*

Proof: Suppose M_ν is dense in itself; that is, every element m_ν of M_ν is the kernel of a convergent net ι_ν defined by $[m'_\nu, m''_\nu]$.

Given any element of the product, $n = (m_1, m_2, \cdots)$ we construct the convergent net I_ν, defined by $[n'_\nu, n''_\nu]$, where

$$n'_\nu \equiv (m_1, \cdots m_{\nu-1}, m'_\nu, m_{\nu+1} \cdots)$$

$$n''_\nu \equiv (m_1, \cdots m_{\nu-1}, m''_\nu, m_{\nu+1} \cdots),$$

such that n is the *kernel*. Hence the full-product is dense in itself.

THEOREM V: *If the full product of a finite number of virtually ordered sets is everywhere dense, then the left-most factor should be everywhere dense.*

Proof: Let $N \equiv V(M_r, \cdots\cdots M_1)$

Let $f' \equiv (m_1', m_2', \cdots\cdots m_r')$,

$f'' \equiv (m_1'', m_2'', \cdots\cdots m_r'')$,

be two elements of N, which are different. If N is dense, then there exists (*i.e.*, we have, for the general case a systematic process for constructing) an element $f''' = (m_1''', m_2''', \cdots)$, such that $f' < f''' < f''$ or $f'' < f''' < f'$. The proof consists in showing that this process fails in the case when all the components except the last are equal in f' and f'', *unless* M_r is dense.

Suppose $m_s' = m_s''$ for $s \leqslant r-1$.

If $f' < f''' < f''$, then $f''' \circ\!\geqslant f'$ and $f''' \leqslant\circ f''$. So, in particular, $m_1''' \geqslant m_1'$ and $m_1''' \leqslant m_1''$. But, since $m_1' = m_1''$, we have $m_1''' = m_1'$ and similarly $m_{r-1}''' = m_{r-1}'$.

Thus. $f' = (m_1', m_2', \ldots . m_{r-1}', m_r')$

$f''' = (m_1', m_2', \ldots . m_{r-1}', m_r''')$

$f'' = (m_1', m_2', \ldots . m_{r-1}', m_r'')$.

Since $f''' > f'$, $m_r''' > m_r'$, and since $f''' < f''$, $m_r''' < m_r''$,

Hence $m_r' < m_r''' < m''$.

Similarly we can prove that. if $f'' < f''' < f'$, then
$m_r'' < m_r''' < m_r'$.

Since m_r', m_r'' are any two elements whatever of M_r, and since m_r''' always lies between them, when $m'_r \neq m_r''$, we see that the assumed systematic process for constructing f''' leads to a systematic process for constructing m_r''', and therefore M_r is everywhere dense.

Ex.: If η is the order type of rational numbers, then $2 \cdot \eta$ cannot be everywhere dense.

THEOREM VI: *If the full product of a finite number or a fundamental series of factors is closed, then each of the factors is closed, provided we can construct an element of each.*

Proof: A virtually ordered set is closed, if it is impossible for a divergent net to exist in it. To prove the theorem, it is sufficient to prove that if we can construct a divergent net in any set M_ν, then we can also construct a divergent net in the full product N.

$$N \equiv V(\cdot \cdot \cdot \cdot \cdot M_\nu, \cdot \cdot \cdot \cdot \cdot, M_2, M_1).$$

Let i_{ν_1}, $i_{\nu_2} \cdot \cdot \cdot \cdot \cdot$ be a divergent net of closed intervals in M_ν such that each is contained in the preceding, and such that, given any element $m_\nu (k)$ of M_ν, we can find an interval i_{ν_r}, $r = r(k)$, such that it is impossible for $m_\nu (k)$ to lie in i_ν.

Let the end-elements of i_{ν_r} be m'_{ν_r}, m''_{ν_r}. Consider the closed intervals I_{ν_r} defined by the end-elements.

$$(m_1, m_2, \cdot \cdot \cdot \cdot \cdot m_{\nu-1}, m_r', \cdot \cdot \cdot \cdot \cdot)$$

$$(m_1, m_2, \cdot \cdot \cdot \cdot \cdot m_{\nu-1}, m_{\nu_r}'', \cdot \cdot \cdot \cdot \cdot)$$

Suppose we are given any element of the product in which the first $\nu - 1$ components differ in some place; then that element cannot lie in any I_{ν_r}.

Next suppose that we are given any element of the product which has only the first $\nu-1$ components equal to $m_1 \cdots\cdots m_{\nu-1}$. Then, for any element $m_\nu(s)$ of M_ν, we can find an interval i_{ν_t}, $t=t(s)$, such that i_{ν_t} does not contain m_{ν_s}. Therefore, I_{ν_t} cannot contain the given element of the product.

Thus, whatever element of the product is given to us, we can find an interval of the sequence I_{ν_r} which does not contain that element. Thus we have constructed a divergent net in the product.

The converse of Th. VI is not true; we have instead,

THEOREM VII: *The full product is closed provided each of the factors possesses the property (A) and, in addition, the property (C) that between any two elements of any factor there exist only a finite number of elements.*

Proof: Suppose there is a divergent net of closed intervals in the product, $i_1, i_2, \cdots$

$$\text{Let } {}_\sigma\alpha' = {}_\sigma a'_1 \; {}_\sigma a'_2 \; {}_\sigma a'_3, \cdots$$

$${}_\sigma\alpha'' = {}_\sigma a''_1 \; {}_\sigma a''_2 \; {}_\sigma a''_3 \cdots$$

be the end-elements of i_σ and, for any σ, let ${}_\sigma\alpha' \leqslant {}_\sigma\alpha''$. We shall now show that, given any positive integer ν, there exists a least positive integer σ_ν, such that ${}_{\sigma\nu}a'_\mu = {}_{\sigma\nu}a'_\mu$, for $\mu \leqslant \nu$, and therefore ${}_\sigma a'_\mu = {}_\sigma a''_\mu$ for $\mu \leqslant \nu$ and $\sigma \geqslant \sigma_\nu$.

We assume that ${}_1a'_\mu = {}_1a''_\mu$ does not hold for every $\mu \leqslant \nu$.

Therefore, ${}_1a'_\mu = {}_1a''_\mu$ for $\mu \leqslant r$ $(r < \nu)$ and ${}_1a'_{r+1} > {}_1a''_{r+1}$.

Then, ${}_\sigma a'_\mu = {}_\sigma a''_\mu = {}_1a'_\mu = {}_1a''_\mu$ for $\mu \leqslant r$, and

$${}_\sigma a'_{r+1} \leqslant {}_1a'_{r+1}, \quad {}_\sigma a''_{r+1} \geqslant {}_1a''_{r+1}.$$

If σ is such that ${}_\sigma a'_{r+1} = {}_1a'_{r+1}$, ${}_\sigma a''_{r+1} = {}_1a''_{r+1}$, then the element ${}_1a''_1 \; {}_1a''_2 \; {}_1a''_3 \cdots {}_1a''_r \; (b''_{r+1}) \cdots\cdots$, where $b''_{r+1} > {}_1a''_{r+1}$, belongs to i_σ.

Since by hypothesis, it is a divergent net, it must be possible to assign h such that i_h does not contain the above element. That is, no such i_σ can lie within i_h, so that at the h^{th} stage and after,

(unspecifiable) one of the above two inequalities (viz. ${}_{\sigma}a'_{r+1} \leqslant {}_{1}a'_{r+1}$, ${}_{\sigma}a''_{r+1} \geqslant {}_{1}a''_{r+1}$) must be an equality; that is, $h\ a'_{r+1} < {}_{1}a'_{r+1}$ or ${}_{1}a''_{r+1} > {}_{1}a''_{r+1}$.

If ${}_{1}a'_{r+1} > {}_{1}a''_{r+1}$, then the reasoning can be repeated for i_h just as we did for i_1, so that the difference between the $r+1^{th}$ components of the end-elements is reduced further. Since, between any two elements, there can exist only a finite number of elements, we ultimately arrive at a stage when the $r+1^{th}$ components are equal.

If ${}_{1}a'_{r+1} = {}_{1}a''_{r+1}$, we can then find a smallest σ_{r+1} for which ${}_{\sigma_{r+1}}a'_{r+1} = {}_{\sigma_{r+1}}a''_{r+1}$, so that ${}_{\sigma_{r+1}}a'\mu = {}_{\sigma_{r+1}}a''\mu$ for $\mu < r+1$. Similarly we construct $\sigma_{r+2}, \sigma_{r+3} \cdots$, such that $\sigma_{\nu+1} > \sigma_{\nu}$ for every ν.

We have then the element ${}_{\sigma_1}a'_1\ {}_{\sigma_2}a'_2\ {}_{\sigma_3}a'_3 \cdots = i\omega$.

Now ${}_{\sigma}\alpha' \leqslant i\omega \leqslant {}_{\sigma}\alpha''$ for every σ, which proves the impossibility of the existence of a divergent net in the product. Hence the product is closed.

§ 4. A Subset of Real Numbers

According to Brouwer, the set A of dually representable[2] real numbers is a subset of the set of real numbers. The set A is not discrete, nor is it ordered. That it is not discrete can be seen from the fact that a real number[3] can be defined which cannot be proved to be either equal to, or different from, zero. That it is also not ordered can be deduced from the fact that in the dual decimal we may not be able to decide which place is occupied by 0 and which place by 1, using the set law alone. But the set A can be virtually ordered, as we shall see presently.

Given a set of integers $a_1a_2 \cdots a_n$, we can make it correspond to a real number which is dually representable by letting the 1's occupy the places with the ranks a_1, $a_1 + a_2$, $a_1 + a_2 + a_3$, $\cdots$ Now, the complex $(a_1, a_2, \cdots a_n)$ may be considered as an element of the full product $V(\cdots, A_2, A_1)$, (where $A_1, A_2, \cdots$ are sets of integers) which is virtually ordered as follows:

Let $f' \equiv (a_1', a_2', \cdots)$; $f'' \equiv (a_1'', a_2'', \cdots)$.

Then, $f' \circ\!> f''$, if there exists an r such that $a_r' \leqslant a_r''$, while $a_\nu' \leqslant a_\nu''$ for $\nu < r$. This relation $\circ\!>$ satisfies $L_1 - L_4$. If we

define its double negation as $>$, then the full product is virtually ordered according to the relation $>$.

This product is everywhere dense by Th. I, it is dense-in-itself by Th. III, and it is closed by Th. VII[2].

I am very thankful to Dr. R. Vaidyanathaswamy for his special course of lectures on Intuitionism, of which this is a direct outcome.

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ON A THEOREM OF GROUP-THEORY CONNECTED WITH A PROBLEM ON PAPER-FOLDING AND WITH SOME OTHER PROBLEMS SOLVED AND UNSOLVED

BY

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A slip of paper $A\ A_m$ may be folded at $A_1, A_2, \ldots, A_{m-1}$ into m congruent parts; so one obtains $2m$ "fields" on the two sides of the paper. The ends A and A_m are joined without twisting the slip. Given n pairs of – say "complementary" – colours a,a'; b,b'; . . . d,d'; the $2m$ fields $A_i\ A_{i+1}$ are be painted of one colour each, but fields on opposite sides of the paper have complementary colours. For slips coloured according to this rule it will be proved:

If the cyclic order of the colours on one side of the slip taken in the clockwise orientation tallies with the anti-clockwise cyclic order of the colours on the other side, then the slip can be folded so that the fields on the innerside are touching two by two, contacting fields having complementary colours.

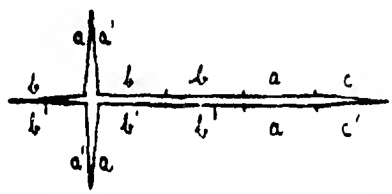
E g. the outerside may be painted

$$a \quad a' \quad b \quad b \quad a' \quad c \quad c' \quad a \quad b' \quad b' \quad a \quad a' \quad b' \quad b$$

therefore the innerside

$$a' \quad a \quad b' \quad b' \quad a \quad c' \quad c \quad a' \quad b \quad b \quad a' \quad a \quad b \quad b'$$

If one starts from the 12th field of the inner side and proceeds to the left, one gets the same cyclic order of the colours as on the outerside from the left to right when starting from the first field. How the slip can be folded according to the proposition. is shown in the figure.



1. The above proposition on slip-colouring is a direct consequence of a theorem on groups and it is nearly equivalent with it.

Consider $a, b, \ldots, d$ as n generators of a free group F and put $a'=a^{-1}, b'=b^{-1}, \ldots, d'=d^{-1}$, then to every sequence of colours there corresponds an element α of F. The inverse element α^{-1} is obtained by interchanging the generators with their inverse elements and taking them from the right to the Left. A cyclic change of the order furnishes a conjugate element. If therefore α corresponds to a sequence of colours satisfying the above conditions, it is conjugate to α^{-1}. On the other hand the element a is not altered if in its representation by generators a pair like $\alpha\ \alpha^{-1}$ is inserted or omitted. If the sequence of colours corresponding to α admits a folding as proposed above, α must be conjugate and therefore be equal to the unit element 1 of F. Hence it suffices to prove that $\alpha \neq 1$ cannot be conjugate to α^{-1}. This statement is the essential part of the following theorem.

If $\alpha \neq 1$ is an element of a free group, then α^m and α^n cannot be conjugate unless $m=n$.

2. Let α^m and α^n be conjugate and β be conjugate to α, then β^m and β^n are conjugate. Hence one can suppose without loss of generality that when α is represented by the generators in the reduced form, the last term in the representation is different from the inverse of the first term. Let q be the length of the reduced representation of α, then the reduced representation of α^m and α^n are

of length $|m|\ q$ and $|n|\ q$ respectively. In each of these representations the last term is different from the inverse of the first one; hence they can be conjugate only if they differ at most by a cyclic permutation and therefore their length must be equal. Thus $|m| = |n|$.

It remains to prove that $m=n$ when $m\neq 0$. Suppose that $m=-n\neq 0$, then $\alpha^m\neq 1$ (since F is a free group) and α^m is conjugate to its inverse.

3. Suppose $\alpha\neq 1$ to be conjugate with α^{-1}. Without loss of generality we can suppose that the first and the last term of the reduced representation of α by the given free generators are non-inverse one to the other. Hence if the terms of that reduced representation are taken in their cyclic order (the 1st and the last being considered neighbours), abutting terms are not the inverse of one another. This cyclic order will be considered now. It is the same for α and for α^{-1} and a cyclic permutation C maps α on α^{-1}. If there are group elements with the required property, then there is one, say α_0, with a minimal length. Let α_0 be generated by generators $a, b, \ldots, d$ where $a^{\pm 1}$ occurs p times $b^{\pm 1}$ occurs q times, $\cdots$, $d^{\pm 1}$ occurs s times; without loss of generality we may suppose that $p \leqq q \leqq \cdots \leqq s$. The length of α is $l=p+q+ \cdots +s$. The terms $a^{\pm 1}$ intersect the remaining cycle of terms into $p' \leqq p$ intervals; equality holds only if the terms $a^{\pm 1}$ are all isolated. If $p' = p = q = l : 2$, the element α_0 depends on a and b only and every $a^{\pm 1}$ and every $b^{\pm 1}$ stands isolated in the cyclic representation; otherwise $s-p>p'$ and there exists therefore an interval between the $a^{\pm 1}$ containing more than one term. Let δ be the word filling such an interval. The cyclic permutation C maps $a^{\pm 1}$ on $a^{\mp 1}$ and therefore maps an interval δ on an interval δ^{-1} and conversely; these intervals have no common element and are separated by other terms containing the term $a^{\pm 1}$. If one replaces every interval δ by b and every δ^{-1} by b^{-1}, one obtains a shorter expression which admits the cyclic transformation C and corresponds to a reduced word. Hence there exists a group-element $\alpha_1\neq 1$ of a shorter length than α_0 which is conjugate to its inverse contrary to our supposition. One must therefore suppose that $p'=p=q=l:2$. So α_0 depends on two generators a and b only and each a-term stands between two b-terms and conversely. Now it will be shown that each b-term stand between an a and an a^{-1}. Suppose there exists a sequence $\Delta = a\ b\ a \cdots a\ b\ a$ of alternating a and b, say k b's and k being the maximal number. By C the word Δ is mapped on $\Delta^{-1} = a^{-1}\ b^{-1}\ a^{-1} \cdots a^{-1}\ b^{-1}\ a^{-1}$ and conversely; thus there exists no

such sequence with more than $2k+1$ terms. Hence neither the Δ nor the Δ^{-1} can overlap in the cyclic order, nor can they abut each $a^{\pm 1}$ standing between two b-terms. If therefore Δ is replaced by b and Δ^{-1} by b^{-1}, one obtains the cyclic order of a reduced word $\alpha_2 \neq 1$ which by the cycle transformation c is mapped on α_2^{-1}. However α_2 is shorter than α_0 contrary to the supposition. Hence b cannot stand between two a's and for the same reason no term can have two equal neighbours. Obviously no reduced word consisting of one or two terms only is conjugate to its inverse. Hence every $a^{\pm 1}$ stands in the cyclic order between b and b^{-1}, every $b^{\pm 1}$ between a and a^{-1}. Therefore the cyclic order of α_0 is either that of $y^m = (ab^{-1}ab^{-1})^m$ or that of y^{-m}. But y^m is not conjugate to y^{-m}; therefore no such α_0 can exist and this finishes the proof.

4. It is of a particular interest to state whether in a group G, where every element $\neq 1$ is of infinite order, none of these elements is conjugate to its inverse element. In this case, G satisfies two necessary conditions for being able to be "ordered". Groups are often considered as topological spaces *i.e.* one introduces in them a notion of "neighbourhood" satisfying the axioms of topology.

These neighbourhoods are supposed to be transformed into neighbourhoods by multiplication with group-elements from the left as well as from the right side. In general a group can be made a topological space in various essentially different ways. One may ask * about the groups which can be made linear topological spaces, *i.e.* in which a transitive relation $\lessgtr$ can be introduced such that when $a \neq b$ *either* $a<b$ and $b>a$, or $b<a$ and $a>b$ hold and $a<b$, $b<c$, $d' \leq d$ imply $a<c$, $d'a<db$, $ad'<bd'$ and $b^{-1}<a^{-1}$. In particular $a<1$, $b<1$ imply $ab<1$, $ba<1$, $a^{-1}>1$, $1>a>a^2>\ldots$ and $k^{-1}ak<1$ for every k. The converse holds for $a>1$, $b>1$. This shows already that an element $\neq 1$ of an ordered group can neither have a finite order nor can it be conjugate to its inverse element. A free group with one generator can be ordered but although these two necessary conditions are satisfied, it is not yet known whether the same proposition holds for free groups with more than one generator. If a group is ordered, all its subgroups are ordered. Now a free group with two generators has subgroups with any finite or enumerable infinite number of generators and obviously the converse holds. Hence it would suffice to decide the question for free groups with two generators.

* see: Proceedings Indian Ac. of Science Vol. XVI (1942) p. 256-263 and Vol. XVII (1943) p. 199-201.

INSTABILITY OF VARIABLE STARS AND THE CEPHEID THEORY OF THE ORIGIN OF THE SOLAR SYSTEM.

BY

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To the common star-gazer, there never occurs any doubt regarding the constancy of the light that reaches him from the age-long stars. There are, however, many stars which have failed to conceal their variability from the practised eye of the astronomer observing them night after night. Such stars, the radiations from which do not remain constant are known as variable stars.

The variable stars fall into two broad groups:—one consisting of those whose intrinsic light actually does not vary; the other includes those which are intrinsically variable. The stars of the former class appear variable to us because they consist of two components, one eclipsing the other at regular intervals. *These are the eclipsing binaries.* The other class consists of stars whose light variation the astronomer has not yet succeeded in explaining fully. This class includes stars varying irregularly and also those whose brightness goes through a cycle in definite periods. These former stars are known as "cataclysmic variables". Among these we find stars of the type R. Coronae Boraealis which become suddenly faint, regaining their former brightness slowly, and stars known as Nova, Supernova, and Subnova which suddenly flare out, come into prominence from invisibility and then gradually fade out until they become again as faint as they were. These are giant stars, but there are also dwarf novae, e.g., the S. S. Cygni stars on the other hand, which have the further interesting characteristic that while the usual novae and supernovae flare up only once in their career and are generally lost afterwards into oblivion, the S.S. Cygni stars repeat their display at regular intervals. Besides these, there are other stars which are bright and faint in an erratic manner including the nebular variables, the irregular variables, with nebular spectra and stars with variable atmospheres.

Next we come to the most important group of variable stars viz., those whose brightness goes through a cycle in definite periods. This group of stars includes:—(a) Long-period variables, with periods ranging from 32 days to 560 days, the best known among which is Mira Ceti, in the constellation of (Ceti) the whale. The variation of its brightness was first noted by Fabricus in 1596 A. D. When brightest, it is a second magnitude star visible to the naked eye, then it fades till in its faintest condition, it is a telescopic star of the ninth or tenth magnitude. When next brightest, it is about of the third magnitude. On an average the period is 330 days; but there is no constancy regarding the value of the period or the actual magnitude in its maximum.

(b) Cepheid variables, of which the typical star is δ- Cephei. The periods of these range from 3 days to 32 days. The variation of these stars are very regular—indeed so regular that their periods have become standard scales in stellar measurements. This group of stars comes first in order of importance among the variable stars, the next being the novae. The third group of the periodic variables is

(c) The Cluster Variables, with periods ranging from 5 hours to 23 hours. The typical among these is R. R Lyrae. These are found in large numbers in globular clusters.

A very important characteristic of the long-period stars is that they are all supergiant stars. Cepheids, too, are giant stars, and a characteristic difference of these from the ordinary stars is in the fact that they are more homogeneous than ordinary stars. In fact, Gamow remarks that the Cepheid variables are in the transition stage of merging into the main sequence of stars and that the variation is due to "instability during transition from the giant branch into the main sequence." (¹)

Among the characteristics of Cepheid variables are:

(1) They are giant stars, but are much more luminous than the normal giants of their type.

(2) As the period increases, the star becomes more and more red.

(3) The logarithms of the period and the magnitude are linearly related.

(4) The rate of brightening is more rapid than the rate of darkening.

(5) There is a marked spherical symmetry. The variation would appear much the same from whatever direction it may be observed.

(6) The spectral lines indicate that there is an approach and recession of the light source, following the same cycle as the variation in brightness. Getting showed that the luminosity attains its maximum value a little before the velocity of approach becomes maximum.

(7) The spectral lines are very sharp and much less diffuse than that which would be expected from a rotational theory of Cepheid variation.

(8) Cepheids are more homogeneous than ordinary stars. The condensation of mass in Cepheids in the centre is less than that in ordinary stars.

Based on these observed facts, theories were formulated to explain Cepheid variation. The original binary star hypothesis in which the variation of light was explained to be due to the motion of a bright primary in a resisting medium round the common centre of gravity of the bright primary and the relatively dark secondary has to be ruled out since it would need the secondary being too near the primary to give a dynamically possible configuration. The fission theory of Jeans, in which we have an envelope of gas surrounding a rotating liquid core near the point of fissional break-up fails to explain spherical symmetry. The theory that holds the ground today is the Pulsation theory conceived by Ritter, Plummer and Shapley and given a mathematical start by Eddington, being later developed by many investigators including

Edgar and Sterne. In this theory we imagine a spherical gaseous equilibrium configuration oscillating symmetrically and radially, the adiabatic condition holding during the oscillation. Naturally the brightness will vary as the gas sphere gets compressed or expanded. But one difficulty arises—we would expect maximum brightness when the star is compressed and we know that as in simple harmonic oscillation the velocity is maximum after a quarter period when the oscillating mass passes through the equilibrium position. But, as we have observed, greatest brightness is in the same phase with the greatest velocity of approach. Further the theory has not yet explained the Period-Luminosity law. Also according to Edgar's calculations the oscillations would die out in an unacceptably short period of the order of 10^4 years. Thus to maintain the oscillations we must have other sources of energy which according to Gamow are " the hydrogen reactions with Li, Be and Bo." These are the observational difficulties. Let us now turn to the dynamical ones.

Eddington, Edgar, and Sterne considered the stars oscillating with small amplitude. For Sterne's three stellar models Bhatnagar was the first to take into consideration the square of the amplitude, and he showed that the consideration of such terms makes the oscillations unstable. This forms the basis of a recent and entirely novel theory of the solar system, by A. C. Banerji [1], who considers the radial oscillations of two stellar models: (1) a homogeneous star (2) a star in which the density varies as the p'th power of the distance, (where p is a positive integer excluding 1 and 3) except in a small finite core of constant density, taking into consideration the square of the amplitude in both the cases. This in both the cases leads to instability.

Imagine now a stable Cepheid pulsating with small amplitude. If now a star passes by, not necessarily too near, then due to the tidal effect the oscillations would become larger and larger. Instability would set in, and hence the Cepheid would break up, resulting in ejection of matter of the Sun's mass, from it. The planets are supposed to form from the ribbon attached to the Sun's mass, and the visiting star is supposed to impart sufficient angular momentum to the planets. Banerji's calculations have shown that a parent Cepheid of about nine times the Sun's mass, the Sun carrying away two-fifths of the energy of the parent Cepheid after formation, would suffice for the formation of the planetary system.

A striking feature of this theory is that it makes the minimum possible number of assumptions and easily explains the usual difficulties. A distant encounter between two stars is definitely much more probable than the accumulation of the special conditions needed for the theory of Jeans and Jeffreys, Russell and Lyttleton, and at the same time it is known that nearly five per cent of all stars are variable. [5] Further for every breaking up of one Cepheid we shall have two planetary systems, one for Sun, the other for the parent Cepheid. These interesting features not only remove the cause of the comment [6] " the solar system had a very narrow escape from never coming into existence " but makes the existence of other planetary worlds definitely more probable.

From the instability of oscillation of slowly rotating models H. K. Sen has shown that a more or less homogeneous star, the instability would lead to fissional break up with the formation of a binary system, while for a centrally condensed core with an extended tenuous envelope, there would be equatorial break up with the formation of the spiral arms. Following Banerji's method, Sen by taking different stellar models has found oscillations stable only for the homogeneous star, which tallies with observation.

So far we have seen the consequences of a steady cumulative effect of tidal forces or rotational momentum on the production of instability in a pulsating star— there is no cataclysmic variation anywhere. Let us now take the case of an intrinsically unstable configuration whose equilibrium is exploded by the lightning visit of an extraneous agency. Thus for example the vast untapped energy of the interior of stars may be released in a cataclysmic penctration of a planet into the stellar interior leading to the blazing of a nova. There are of course better theories to explain the formation of a nova—thus suppose that a dwarf is passing through an envelope of nebulous matter; it goes on gathering mass round it like a blanket with the resultant storage of energy in the interior. This process obviously cannot go on indefinitcly—a time comes— say in a century—when the whole blazes into a nova. Nebulous envelopes have been observed round novae that have burst. The abundance of dwarfs, and the nebulae and the observed frequency of novae supply good confirmation of the theory.

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NUMERICAL NIGHTMARES *

BY

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If you wish to tempt your friend into a foolish answer, you may propose to him one of those problems, ancient or modern, which will test his appreciation of the frightful rapidity with which certain numbers grow. Ten to one, he will fail miserably and will leave you in a hurry on the pleasant errand of making others look as silly as he himself did.

Double, double, never mind trouble

Perhaps you have heard the story of the inventor of the game of Chess who when asked by the King of his land to name his own reward said :—" O Great Monarch whose generosity is the envy and the despair of the other monarchs in the seven worlds above and the seven worlds below, grant me but this boon. For the first square in yonder chessboard give me one grain of paddy, for the second square two grains, for the third square four grains, and so on doubling each time till all the 64 squares are exhausted." The King was disappointed at this apparently modest request and offered to begin with a bag instead of a single grain, but the inventor said he would be satisfied with the single grain. And so the King passed the necessary orders to his Minister in Charge of Charities who, in his turn, commanded the State Computer to make the necessary calculations. After figuring it all out, the Computer declared that it amounted to $2^{64}-1=18446744073709551615$ grains and that there was not enough paddy in the land to carry out the order; for if it were all spread uniformly over the 10,000 square miles of the kingdom, it would have to be piled so high that the tallest temple gopurams would be submerged. The story does not state how the King met the situation. Perhaps he bestowed on the inventor half his kingdom and the hand of his daughter *Dhanyasindhu* (Ocean of Grain) marriage.

My next example of doubling numbers should endear me to the missionaries of birth control and make their patron Saint Malthus sit up in his grave—if he should be still there. If a man has two

* This is adapted from the text of a Radio Talk broadcast from the Madras Station of All India Radio under School Broadcasts, and is published with the permission of the A. I. R. Madras

children and each of them two children and so on in succession, what will be the total progeny after say 70 generations? An easy calculation will show that in 32 generations they will amount to the present human population of the globe. 20 generations later they will have "multiplied and replenished the earth" to the extent of *having just enough standing room* only, with a modest square foot per person. Thereafter we shall have to arrange them in columns piled one over the other. In 20 generations more each of these human columns will have reached a height of a thousand miles. It would be cruel to carry on these calculations farther.

Powerful Numbers

If your friend and victim has not meanwhile run away or murdered you peacefully, you may next ask him whether he has any idea of the biggest number which may be formed with 4 figures. He will answer that it is 9999, and will be quite right if the figures are to be written down as usual. But, if they could be used as powers unimaginably big numbers which stagger the imagination may be written down. Thus with two nines we have 9^9 which means $9 \times 9 \times 9 \times 9$...nine times. This number which I shall call A has the value 387,420,489. With three nines we may form 9^{9^9} which means 9^A. I shall call this number B. It begins with 428,124,773,175,747,048,036, 987,115 and ends with the digits 89. I hope my considerate reader will not insist on all the figures being given for there are about 300 million of them and if printed in full it would not only occupy the whole of this issue of *The Mathematics Student* to the exclusion of all other matter, but also require all such issues for the next 300 years, and I am afraid you may consider the article somewhat long! Written on a strip of paper so as to be readable, it would require a strip a thousand miles long, while printed in book form it would make a small library of 42 volumes similar to those of the *Encyclopaedia Britannica.* All this using only three nines. With four nines we have the number $9^{9^{9^9}} = C$ which is the same as 9^B. If we could compress all the 42 volumes referred to above into a grain of sand and close-pack such grains so that they formed a sphere extending up the farthest nebulae faintly visible in our biggest telescopes, we shall still have a library which is hopelessly inadequate for expressing this number C. In fact even if this huge sphere were again compressed to the size of a grain of sand and again such

grains close-packed to the size of the big sphere, we shall not have made a serious attempt to have a library adequate for this number. But why attempt the impossible. As the Upanishadic Seers have it, "words return back baffled along with the mind without having reached it." It is a definite number all the same, and to be condemned to its calculation is the equivalent of the Eternal Hell in the mathematician's *inferno*, reserved only for those who have committed the five deadly sins against mathematics (such as division by zero etc.).

Kasner's Googol

Prof. Edward Kasner of the Columbia University has coined the word "googol" for 1 followed by a hundred zeros. It is a pigmy compared with the number $9^{9^{9}}$, but is itself so huge that we shall never require it in any counting or measurement. The largest figure in finance was probably the total number of paper marks in circulation in Germany at the peak of the inflation when it was said you could go to the market with the money carried in a basket and bring the vegetables in your purse. Even so it was a figure of only 20 digits and so, very much less than a googol. The total number of sand particles on the Madras Beach—including, of course both the Tamil and Andhra sands in the reckoning—would be a still smaller figure. The total number of drops of water in the ocean is less than a googol, being a miserable figure with about 28 digits. If all the matter not only in the Earth but throughout the Stars, the Milky Way and all the Nebulae in the Universe were converted into atoms and then into electrons, how many would they amount to? Eddington has calculated this by making use of Einstein's Theory of Relativity and his estimate is a figure with 80 digits. So you see we have still not reached the googol. How much vaster is $9^{9^{9}}$, which contains nearly three hundred million digits while the googol has only a hundred and one!

The next time you write to thank anyone, send him $9^{9^{9}}$ thanks, or if you are of a miserly temperament, send him at least a googol of them. To convey "a thousand thanks" when these larger consignments cost no more is a wanton waste of the milk of human kindness.

ON THE DIFFERENTIAL EQUATION $f^r(x) = f(1/x)$

BY

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DR. LUDWICK SILBERSTEIN *, has solved the hystero-differential equation $f'(x) = f(1/x)$. In this paper the solution of the hystero-differential equation $f^r(x) = f(1/x)$ is discussed and that of

$$f''(x) = f(1/x)$$

is given as an illustration.

1. Our equation is $\qquad f^r(x) = f(1/x) \qquad$ (1)

Let us assume that, $\qquad f(x) = x^m + \lambda x^n \qquad$ (2)

where λ is independent of x. Substituting, we have

$$m(m-1)\ldots(m-r+1)x^{m-r} + \lambda n(n-1)\ldots(n-r+1)x^{n-r} = x^{-m} + \lambda x^{-n} \quad (3)$$

from which we obtain,

$$\left.\begin{aligned} &m+n=r;\ \lambda = m(m-1)\ldots(m-r+1) \\ &\lambda n(n-1)\ldots(n-r+1) = 1 \end{aligned}\right\} \quad (4)$$

By eliminating λ and n we get

$$m(m-1)^2\ldots(m-r+1)^2(m-r) = (-1)^r \quad (5)$$

The equation (5) may be re-written.

$$\left.\begin{aligned} &[y+(r-1)]^2[y+2(r-2)]^2\ldots[y+\tfrac{1}{2}(r-1)\cdot\tfrac{1}{2}(r+1)]^2 y = (-1)^r \quad (r \text{ odd}) \\ &[y+(r-1)]^2[y+2(r-2)]^2\ldots[y+(\tfrac{1}{2}r)^2]y = (-1)^r \quad (r \text{ even}) \end{aligned}\right\} \quad (6)$$

where $y = m^2 - mr$.

Therefore we see that the values of m depend upon the solution of an equation of rth degree in y. Let $y = y_1, y_2, \ldots y_r$ be the solutions of the equation (6). (6)

Now if $y = y_s$ be a solution of (6) we get $m^2 - mr = y_s \qquad$ (7)

Let m_{1s}, m_{2s} be the solutions and $\lambda_{1s}, \lambda_{2s}$ be the corresponding values of λ from (4): Thus we get two solutions corresponding to $y = y_s$ where $m_{1s} + m_{2s} = r$. It can be easily shown that these two solutions are identical since $\lambda_{1s}\lambda_{2s} = 1$.

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* Silberstein *Phil. Mag.* Vol. 30 (1940). 185–87

$\therefore$ The complete solution of the equation $f'(x) = f(1/x)$ may be written

$$f(x)=\sum a_s U_s (s=1, 2, 3 \ldots r); \; U_s = x^{m_{1s}} + \lambda_{1s} x^{m_{2s}}, \ldots \quad (8)$$

where the m, s are the solutions of the equation $m^2 - mr = y_s$, and a's are the r arbitrary constants.

2. *Solution of $f''(x)=f(1/x)$*

In this case equations (4), (6) and (7) reduce to

$$\left.\begin{aligned} \lambda &= m^2 - m; \\ m^2 - 2n &- y; \\ y^2 + y - 1 &= 0; \end{aligned}\right\} \quad (9)$$

From the last equation we obtain,

$$y=(-1\pm\sqrt{5})/2. \quad (10)$$

$\therefore$ either, $m^2 - 2m - \dfrac{-1+\sqrt{5}}{2} = y_1$ or $m^2 - 2m = \dfrac{-1-\sqrt{5}}{2} = y_2$. (11)

from which we obtain,

$$\left.\begin{aligned} m_{11} &- \tfrac{1}{2}(2+\sqrt{2+2\sqrt{5}}) \\ m_{21} &= \tfrac{1}{2}(2-\sqrt{2+2\sqrt{5}}) \\ m_{12} &- \tfrac{1}{2}(2+\sqrt{2-2\sqrt{5}}) \\ m_{22} &= \tfrac{1}{2}(2-\sqrt{2-2\sqrt{5}}) \\ \lambda_{11} &= \tfrac{1}{2}(1+\sqrt{5}+\sqrt{2+2\sqrt{5}}) \\ \lambda_{12} &= \tfrac{1}{2}(1-\sqrt{5}+\sqrt{2-2\sqrt{5}}) \end{aligned}\right\} \quad (12)$$

Using these values of m and λ we get the complete solution of the differential equation.

I am indebted to Mr. G. R. Seth, lecturer Hindu College Delhi, for various suggestions and improvements in the preparation of this paper.

NOTES AND DISCUSSIONS

Cylindrical projection and rolling

1. The axes being rectangular, the two generators of the right circular cylinder: $y^2+z^2-z=0$, lying in the XOZ plane are given by the equations:

$$y=0,\ z=0;\ \text{and}\ y=0,\ z=1.$$

The former is the X-axis. We shall call the latter the 'second' generator.

If $P(x, y, 0)$ be any point in the XOY plane and the perpendicular from P to the second generator meet the cylinder in the points Q and R of which the point R is on the second generator, then Q shall be called *the projection of P on the cylinder.*

It is easy to show that the co-ordinates of Q are $\left(x, \frac{y}{1+y^2}, \frac{y^2}{1+y^2}\right)$; so that if the point P traces the curve: $y=f(x)$, $z=0$; then the point Q traces the curve:

$$z=yf(x),\ y^2+z^2-z=0$$

Moreover, the projection of the point $P_1(x, 1/y, 0)$ on the cylinder is the point $Q_1\left(x, \frac{y}{1+y^2}, \frac{1}{1+y^2}\right)$, which is the reflection of the point Q in the plane $z=\frac{1}{2}$. It would thus be seen that the projections of the two curves: $y=f(x)$, $z=0$; and $y=1/f(x)$, $z=0$ on the cylinder are reflections of each other in the plane $z=\frac{1}{2}$.

2. The cylindrical projection of the curve: $y=f(x)$, $z=0$, may be obtained by rolling or wrapping round the cylinder, the curve:

$$y=\tan^{-1} f(x), \qquad z=0,$$

in such a manner that the axis of x remains fixed.

"Rolling" transforms the point $(x, y, 0)$ in the XOY plane, into the point $(x, \sin y \cos y, \sin^2 y)$ on the cylinder.

The curve: $$y=\phi(x),\ z=0,$$

is thus transformed into the curve:

$$z=y\tan\phi(x),\quad y^2+z^2-z=0$$

by rolling.

3. In particular, the projection of $y = \tan x$, $z = 0$, is obtained by rolling round the cylinder. the straight line $y = x$, $z = 0$. It is noteworthy that while the tangent graph is discontinuous, its projection on the cylinder is a continuous curve.

The projections of the sine and cosecant; and the cosine and secant graphs are equally interesting. They are not only reflections of each other in the plane $z = \frac{1}{2}$, but lie completely on either side of it on the cylinder, touching each other on the two generators of the cylinder lying in the plane $z = \frac{1}{2}$.

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Notes on Ellipse

This note establishes certain simple relation and constructions (associated with ellipse) that the writer has found out. All quantities whose constructions have been given are associated with a point whose radius vector (r_1) is known.

Let P be a point on an ellipse (axes a, b), having S_1, S_2 as foci, c as centre, and TPT' as tangent at P.

Let $r_1,\ r_2,\ r = S_1P,\ S_2P,\ CP$;

$p_1,\ p_2,\ p =$ perpendiculars on TPT' from S_1, S_2, C;

$\alpha = \angle TPS_1 = \angle T'PS_2$; $f^2 = r_1 r_2$

We define ω by $f = a \sin \omega$. ... (A)

It is easily seen that $2p = p_1 + p_2$.

$\therefore \quad 2p = p_1 + p_2 = r_1 \sin \alpha + r_2 \sin \alpha = 2a \sin \alpha$

$\therefore \quad p = a \sin \alpha$ ... (1)

From the pedal equation of the ellipse we have

$$\frac{b^2}{p_1^2} = \frac{2a}{r_1} - 1 = \frac{r_2}{r_1} \qquad \ldots \quad (B)$$

Hence $2p = b\,\{\sqrt{(r_1/r_2)} + \sqrt{(r_2/r_1)}\} = b\,\dfrac{r_1 + r_2}{\sqrt{r_1 r_2}} = \dfrac{2ab}{f}$

$\therefore \quad pf = ab$ ... (2)

Substituting from (A) and (1) in (2) we get

$$p \sin \omega = b \qquad \dots \quad (3)$$

$$f \sin \alpha = b \qquad \dots \quad (4)$$

Again, by differentiating logarithmically the result (B) we have

$$-2\frac{dp_1}{p_1} = \frac{dr_2}{r_2} - \frac{dr_1}{r_1} = -dr_1\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \text{ since } dr_1 + dr_2 = 0.$$

$$\therefore \quad 2\frac{dp_1}{p_1} = \frac{2a\, dr_1}{r_1 r_2} \quad \therefore \quad \rho = r_1 \frac{dr_1}{dp_1} = \frac{r_1 r_2}{a} \cdot \frac{r_1}{p_1}$$

$$i.e. \quad \rho = \frac{r_1^2 r_2}{a} \cdot \frac{1}{b}\sqrt{\frac{r_2}{r_1}} = \frac{(r_1 r_2)^{3/2}}{ab} = \frac{f^3}{ab} \qquad \dots \quad (5)$$

Constructions:—

With $AB = 2a$ as diameter draw a semicircle. Let O be its centre. Cut off AN on AB making $AN = r_1$, then $BN = r_2$.

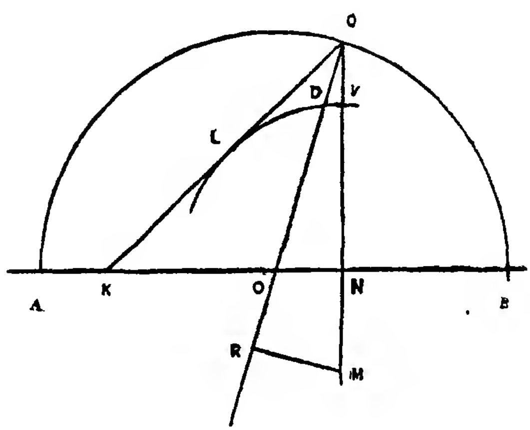
At N erect perpendicular to AB cutting the circle at Q. Then by elementary geometry $NQ^2 = AN \cdot BN = r_1 \cdot r_2$ $\therefore$ $NQ = f$ ... (i)

$$\therefore \quad \text{Sin } \angle NOQ = \frac{NQ}{OQ} = \frac{f}{a} = \text{Sin } \omega \quad \therefore \quad \angle NOQ = \omega \qquad \dots \quad \text{(ii)}$$

With centre N and radius b draw a circle cutting NQ at V. Draw QL tangent to this circle from Q.

$$\text{Then } \sin \angle NQL = \frac{LN}{NQ} = \frac{b}{f} = \sin \alpha, \text{ by (4)}$$

$\therefore$ $\angle NQL = \alpha$ ... (iii)



Draw VD perpendicular to VN cutting OQ at D.

$\therefore$ $OD \cdot \sin \omega = b$.

Hence by (3) we have $OD = p$ . . . . . (*iv*)

Produce QL to meet AB at K.

Cut off $QM = KQ$ on QN produced, and from M drop the perpendicular MR on QO (produced if necessary) then, using (A), (1), (2) we have

$$QR = QM \cdot \cos \angle OQN = KQ \cdot \sin \omega$$

$$= \frac{f}{\sin \alpha} \cdot \sin \omega = \frac{f}{a \sin \alpha} \cdot a \sin \omega = \frac{f^2}{p} = \frac{f^3}{pf} = \frac{f^3}{ab} = \rho \qquad \ldots \quad (v)$$

Also

$$\begin{aligned} r_1^2 &= r^2 + a^2 e^2 - 2aer \cos \theta \\ r_2^2 &= r^2 + a^2 e^2 + 2aer \cos \theta \end{aligned} \quad \text{where } \theta = \angle S_1CP.$$

Adding $r_1^2 + r_2^2 = 2r^2 + 2a^2e^2 = 2r^2 + 2a^2 - 2b^2$.

Adding $2f^2$ or $2r_1 r_2$ to both sides, we have

$$(r_1 + r_2)^2 = 2r^2 + 2a^2 - 2b^2 + 2f^2$$

or $\quad 4a^2 = 2r^2 + 2a^2 - 2b^2 + 2f^2$

$\therefore \quad r^2 = a^2 + b^2 - f^2 = a^2 - a^2 \sin^2 \omega + b^2 = a^2 \cos^2 \omega + b^2$

But $a \cos \omega = OQ \cdot \cos \angle QON = ON$

$\therefore \quad r^2 = ON^2 + VN^2 = OV^2 \quad$ i.e. $\quad OV = r \qquad \ldots \quad (vi)$

GAGANBIHARI BANDYOPADHYAYA.

A Neglected equation in Analytical Conics

A writer in the *Mathematical Gazette* * speaks of the equation to the chord of

$$s \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \dagger$$

with a given mid-point as a much neglected equation. Curiously enough he approaches the equation through another even more neglected. The latter equation, that of the chord joining two given points on a conic, furnishes a ready interpretation of the "general equations" connected with $s = 0$ and will be found helpful in a rapid preliminary survey of the conic regarded merely as a curve of the second degree. The indifference to this equation shown by our text-books on Analytical Conics is surely not merited.

Let us write

$$s_{pq} = ax_p x_q + h(x_p y_q + x_q y_p) + \cdots + g(x_p + x_q) + \cdots + c = s_{qp}$$

* February 1942 *Gazette*, p. 51.

† Needless to say, the arguments presented in this note are applicable to any system of homogeneous co-ordinates provided C in paragraph three is replaced by Cl where $l = 0$ represents the line at infinity.

and denote by s_{pp}, s_p respectively the expressions obtained from s_{pq} by changing the suffix q into p and dropping the suffix q.*

The equation to a straight line can be related to the conic $s=0$ and to two given points 1 (x_1, y_1) and 2 (x_2, y_2) by writing it in the form $As_1+Bs_2+C=0$. If the points are on both the straight line and the conic, $As_{11}+Bs_{21}+C=0=As_{12}+Bs_{22}+C$ and $s_{11}=0=s_{22}$ so that $C/A = C/B = -s_{12}$. Hence the equation to the chord joining the points 1 and 2 on $s=0$ is

$$s_1+s_2=s_{12}.$$

By letting the point 2 tend to the position of 1 in the above, we obtain the equation to the tangent at 1 to the conic in the form $2s_1=s_{11}$ or $s_1=0$.

If the points $2(x_2, y_2)$ and $3(x_3, y_3)$ on the conic are connected by a chord whose mid-point is $1(x_1, y_1)$, the equation to the chord is $s_2+s_3=s_{23}$ which, after replacing s_2+s_3 by $2s_1$, becomes $2s_1=k$ (constant) $=2s_{11}$. Thus the equation to a chord of $s=0$ can be expressed in terms of the co-ordinates of its mid-point 1:

$$s_1 = s_{11}.$$

Let now 1 be any fixed point dividing a variable chord through it and through 2, 3 on the conic in the ratio $\lambda:1$. Then, remembering the relation between the co-ordinates of 1 and those of 2, 3, we can write the equation $s_1=0$ in the form $(s_2+\lambda s_3)/(1+\lambda)=0$ and show that $s_1=0$ always passes through the intersection of the tangents $s_2=0$ and $s_3=0$. Hence the polar of 1 w.r.t. $s=0$, defined in the usual manner, is given by

$$s_1 = 0.$$

The Harmonic Property of Pole and Polar can be readily established by pursuing further the argument in the last paragraph. Let $s_1=0$ divide the chord joining 2 and 3 in the ratio μ. Then $\mu = -s_{12}/s_{13}$ which, in virtue of the relation $s_1\equiv(s_2+\lambda s_3)/(1+\lambda)$, leads to

$$\mu = -\frac{s_{22}+\lambda s_{32}}{s_{23}+\lambda s_{33}} = -\lambda.$$

The reader will no doubt recall that the shortest method of obtaining the pair of tangents from (x_1, y_1) to $s=0$ takes as starting-point the equation $s+ks_1^2=0$. This method and our discussion both owe their simplicity to a systematic exploitation of the formal relations between s_{pq}, s_p and s.

Christian College, Madras.

C. T. RAJAGOPAL

* This notation is due to A. Robson. (February 1942 *Gazette*, p. 49.)

ANNOUNCEMENTS AND NEWS

The following gentlemen have been admitted as members of the Indian Mathematical Society.

S. A. Hamid Esq , M.A. (Cantab), P. E. S., Government College, Lahore.

D. R. Jain Esq., M.A., Government College, Hoshiarpur.

Gian Chand Esq., M.A., ,, ,, ,,

Dr. S. Chowla, Ph D. (Cantab), Government College, Lahore.

A. M. Krishnamurti Esq., M.A., National College, Trichinopoly.

Arrangements are in progress for the Thirteenth Biennial Conference of the Indian Mathematical Society which is to meet at Annamalainagar on the 28th, 29th and 30th of December 1943. The dates have been so fixed that the delegates proceeding to the Indian Science Congress at Trivandrum, will be able to halt at Annamalainagar *en route*, participate in the activities of the Conference, and still be in time for the Science Congress. All those desirous of attending the Mathematical Conference are requested to communicate with the Local Secretary, Dr. A. Narasinga Rao, Dean, Faculty of Science, Annamalai University, Annamalainagar, stating the date of arrival and the kind of food they will require.

The "Mathematical Exposition" will be opened on the afternoon of the 28th December. Besides models, pictures, charts and other material intended to illustrate interesting results in Mathematics and the richness and variety of its applications to life situations, there will be a "Book Section" for exhibiting text-books on Collegiate and Higher Mathematics and its applications. All books, intended for the exhibition may be sent to the Local Secretary to reach him at least a week before the Conference. Those willing to send charts, diagrams or models should communicate with the Local Secretary.

An appeal for pictorial and other material for the Mathematical Exposition has met with a very generous response from the U. S. A. The authorities of the popular Mathematical Journal, *Scripta Mathematica* have been pleased to present a large number of portraits, of Mathematicians, Physicists and Philosophers as also a complete set of their "Scripta Mathematica Library" publications. Messrs. Simon and Schuster have presented their fascinating books: Bell: *Men of Mathematics*, and Kasner and Newman: *Mathematics and the Imagination*, while Messrs. H. G. Lieber have sent four delightful booklets on Relativity, Non-Eucledean Geometry, Algebra

and Galoisian groups which mark a new era in simple and attractive presentation of mathematical topics.

The Editors of the Mathematical Reviews, Brown University Providence, R. I., U. S. A., are finding it difficult to procure for purposes of review, research papers published in India. They request the cooperation of Indian Mathematicians by sending them promptly reprints of their papers.

The National Academy of Sciences, India, at its 12th Annual Session held at Allahabad awarded the Gold Medal offered by Dr. Panna Lal, Adviser to the Governor of the U. P. to Dr. Ram Behari, Professor of Mathematics, St. Stephen's College, Delhi, for his group of papers on "Differential Geometry" which have been assessed as the best papers in Mathematics published in the Proceedings of the Academy during the last five years.

Non-Solar Planetary Systems: Till recently there was no evidence of planetary systems belonging to stars other than our Sun, but the *Observatory* of May 1943 gives details regarding two cases of visual binaries where parallax observations have led to the inference that a third invisible companion exists sufficiently small to be classified as a planet. In the system 61 *Cygni* (period 720 years) the deviations from the Keplerian motion have a period of 4·9 years, and the invisible companion must be of mass 0·016 ⊙ with a semi-major axis of 2·4 astronomical units and a highly eccentric orbit ($e = \cdot 7$). In the other case 70 *Ophiuchi* (period 88 years), the deviations have a period of 17 years and the disturbing planet should have a mass 0·012 ⊙ or ·008. according as it is considered as belonging to one or the other of the two components of the binary system. In both cases the perturbing mass is less than 1/9th of the mass of the lightest known star (*Kruger* 60B = 0·14 ⊙).

Prof. Earl Raymond Hedrick of the Brown University well known author of text books, died in February 1943.

MATHEMATICAL GREETINGS TO THE NEW YEAR

O Mother Earth, greetings to you with all hearty good wishes, ere you begin your next course on a long elliptic journey of 292 million miles. This is your one thousand nine hundred and forty third course in the heavens, reckoned from the date when the Son of God went back to the Father in Heaven.

Mortal man keeps count of the successive steps of your career by an elaborate machinery which he calls the 'Calendar,' printing it year in and year out with dim awareness of your mathematical regularities. Wise men know that the next year of grace 1943 is identical in dates and week-days with those of several years that have gone before and several others yet to come.

In illustration thereof, it may be pointed out that the calendars of the years 1909, 1915, 1926, 1937, 1943, 1954, 1965, 1971, 1982, 1993, 1999 are identical! Note that the coming year is the fifth spoke in the wheel of the 6–11–11 cycle.

Ye aspirants to centenarianism realise the value of the New Year calendar till the penultimate year of this century.

Ye men of the year of grace 1937 that have preserved untarnished the calendar of that year celebrate its re-incarnation in the New Year.

Ye number-intoxicated children of Mother Earth, rejoice in the birth of the New Year, and propitiate her in her manifold mathematical manifestations as the glorious product of two primes (29×67) the illustrious union of three $(643+647+653)$, the powerful combination of a plane and a solid number (40^2+7^3), the harmonious whole of four beautiful squares $(1^2+6^2+15^2+41^2)$, or a square and two cubes $(10^2+8^3+11^3)$ the lawful predecessor of the sum of two cubes (12^3+6^3) and the product of a cube and a fifth power $(2^3 \times 3^5)$.

Ye philosophers, contemplate on this fresh section of the Eternal Element of Time, and surrender your all in humble homage to the DIVINE IMMINENCE.

A. A. K.

GLEANINGS

"Mathematics, to my way of thinking, is the most general of all subjects. Everything else is more special. There is no school subject that has a richer profusion of applications. There is nothing that travels over the whole domain of human knowledge as does mathematics. There is no surer way to unlock all sorts of doors than mathematics. It won't get you all the way, but it will get you into places where you can't enter by any other method. It will supplement all other types of investigations and help get at profounder truths than will be possible without its aid."

Prof. HOTELLING.

A knowledge of mathematics is one of the foundations on which the operations of modern armies are based. It cannot take the place of study of tactics and strategy, but now that armies are so thoroughly mechanized. It is impossible to visualize their successful operation unless the officers and men of which they are composed have a through groundwork in mathematical training.

Major BARRETT.